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LVII. *The Selectivity of Photometric Integrators, with particular reference to the Photometry of Gaseous Discharge-Tubes.* By H. BUCKLEY, M.Sc., F.Inst.P. (National Physical Laboratory) *.

ABSTRACT.

THE paper gives a theoretical treatment of the effect of selectivity of photometric integrators in producing errors in the determinations of the luminous outputs of incandescent lamps and gaseous discharge lamps. The dependence of the error on the method used is investigated. It is shown that in suitable easily realized circumstances measurements of the luminous outputs of neon, sodium, high-pressure and low-pressure mercury lamps should not be in error by more than 1 per cent. due to the selectivity of the photometric integrator.

Introduction.

THE effect of selective absorption in the white paint used in photometric integrators is to produce an error in the value obtained for the luminous output of a test source which has a distribution of energy in its spectrum different from that of the standard source with

* Communicated by the Author.

which it is compared. A considerable amount of attention has been paid to the subject by workers in photometry, including Cady ⁽¹⁾, Dziobek ⁽²⁾, Fox ⁽³⁾, and in particular by Walsh and Barnett ⁽⁴⁾. The error is primarily due to the fact that in these circumstances the reflexion factors of the walls of the integrator are different in the two cases.

Unless something is known of the selective properties of the paint used in the walls of the integrator, it is impossible to deduce any information as to the magnitude of the errors produced in any given circumstances.

Two grey-body radiators can, however, be operated so that they colour-match. When one of them is placed inside the integrator and the radiation from the diffusing window in the wall is compared with that from the other source, it is usually found that the colour-match is destroyed. It is an experimental fact, however, that for most white paints used in integrators the colour-match can be restored if the voltage on the lamp outside the sphere is altered. Usually it has to be lowered so that the effect of selectivity is to reduce the colour-temperature of the lamp in the integrator. Fox ⁽³⁾ has described a method of neutralizing the "reddish filter" effect of the paint by inserting an actual bluish filter in the front of the sphere-window to restore the colour-match.

If it now be assumed that the colour-match obtained with one lamp inside the integrator implies identity of energy distribution in the visible portions of the spectra of the two sources, as well as identity in colour, the assumption can be used in a quantitative manner to determine the errors resulting from the selectivity.

The errors produced in the comparison of grey-body radiators operating at different colour-temperatures have been dealt with by Walsh and Barnett ⁽⁴⁾. These workers, using the experimental fact and assumption mentioned above, regarded the selectivity of the sphere paint as being equivalent to a colour-filter whose transmission τ_λ at any wave-length λ is given by

$$\tau_\lambda = k e^{-\frac{c_2}{\lambda} \left(\frac{1}{T_0'} - \frac{1}{T_0} \right)},$$

where k is a constant, T_0 is the colour-temperature of a source outside the sphere, and T_0' is the colour-temperature

of the radiation from the window of the integrator when the same source is placed inside. This equation is, of course, deduced from Wien's Law.

Walsh and Barnett calculated the error introduced by this amount of selectivity in the comparison of sources of various colour-temperatures with a standard source of colour-temperature 2400° . This was done by computation of the integral transmission ratios of the filter τ_{λ} for light of various colour-temperatures. In particular, they found that if the equivalent filter τ_{λ} were such as to reduce the colour-temperature of a source at 2400° by 150° , the error in comparing a tungsten vacuum lamp of colour-temperature 2400° with a standard carbon lamp of colour-temperature 2000° was about 0.8 per cent.

Walsh and Barnett limited their investigation to the consideration of the errors produced in the photometry of grey radiators, *i.e.*, tungsten and carbon lamps. It is, however, possible to extend the investigation to deal with sources giving more or less homogeneous radiations, such as the gaseous discharge lamps which are now being used. It is also possible, by the use of the Crova wave-length⁽⁵⁾ and the mean effective wave-length⁽⁶⁾, to give the results of Walsh and Barnett in a more explicit form.

The Effect of Selectivity in the Photometry of Grey Radiators.

In order to clarify the argument, it will be convenient to describe briefly the method by which the luminous outputs of two sources may be compared in a spherical integrator.

The source to be measured, whose luminous output is L lumens, and a standard source of luminous output L_s lumens, are placed successively in the centre of the integrator, as shown in fig. 1. The inside of the integrator is painted with a white diffusing paint whose diffuse reflexion factor is r_{λ} for light of wave-length λ . A small diffusing window, whose diffuse transmission ratio is t_{λ} , is placed in the wall of the integrator. A small whitened screen S , placed between the centre of the integrator L and the window W , prevents direct light from either of the sources from reaching the window. Thus W is

illuminated by light which has undergone multiple reflexions at the walls of the integrator.

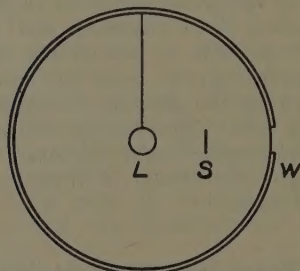
If the sources L and L_s have colour-temperatures T and T_s , and emit amounts of energy given by $aE(\lambda T)$ and $a_s E(\lambda T_s)$ at wave-length λ , where $E(\lambda T)$ is Wien's function, and a and a_s are constants, then the brightness of the window viewed from outside will be given by

$$B = pa \int E(\lambda T) \frac{r_\lambda t_\lambda}{1 - r_\lambda} K_\lambda d\lambda$$

when the test-source is in the integrator, and by

$$B_s = pa_s \int E(\lambda T_s) \frac{r_\lambda t_\lambda}{1 - r_\lambda} K_\lambda d\lambda$$

Fig. 1.



Photometric integrator.

when the lumen standard is in the integrator, where K_λ is the relative visibility of radiation of wave-length λ , and p is a constant depending on the size of the integrator.

If we replace the terms r_λ and t_λ involving the selective properties of the integrator paint and window by the equivalent filter

$$\tau_\lambda = ke^{-\frac{c_2}{\lambda} \left(\frac{1}{T_0'} - \frac{1}{T_0} \right)} = ke^{-\frac{mc_2}{\lambda}},$$

where $m = \frac{1}{T_0'} - \frac{1}{T_0}$ is a constant defining the selectivity,

we have as the observed ratio of the brightnesses of the window in the two cases

$$R_0 = \frac{B}{B_s} = \frac{a \int \tau_\lambda E(\lambda T) K_\lambda d\lambda}{a \int \tau_\lambda E(\lambda T_s) K_\lambda d\lambda}.$$

If $m=0$, i. e., if the equivalent filter is non-selective, we have

$$R = \frac{a \int E(\lambda T) K_{\lambda} d\lambda}{a_s \int E(\lambda T_s) K_{\lambda} d\lambda} = \frac{L}{L_s}.$$

Hence

$$\begin{aligned} \frac{R}{R_0} &= \frac{\int \tau_{\lambda} E(\lambda T_s) K_{\lambda} d\lambda}{\int E(\lambda T_s) K_{\lambda} d\lambda} \cdot \frac{\int E(\lambda T) K_{\lambda} d\lambda}{\int \tau_{\lambda} E(\lambda T) K_{\lambda} d\lambda} \\ &= \frac{\tau_{\Sigma}(T_s)}{\tau_{\Sigma}(T)}, \end{aligned}$$

where $\tau_{\Sigma}(T_s)$ and $\tau_{\Sigma}(T)$ are the integral transmission ratios of the equivalent filter τ_{λ} for radiations of colour-temperature T_s and T respectively. Since, however, τ_{λ} is of the form which converts a colour-temperature T_0 in the Wien expression to a colour-temperature T_0' , it will change colour-temperatures T and T_s to T' and T_s' respectively in accordance with the following relations :

$$\frac{1}{T_0'} - \frac{1}{T_0} = \frac{1}{T'} - \frac{1}{T} = \frac{1}{T_s'} - \frac{1}{T_s} = m.$$

Its integral transmission ratios for radiations of colour-temperatures T and T_s are, therefore, equal to its actual transmission ratios at the mean Crova wave-lengths λ_c , for the two colour-temperature intervals T to T' and T_s to T_s' .

Thus

$$\frac{R}{R_0} = \frac{\tau_{\lambda_c}(T_s T_s')}{\tau_{\lambda_c}(T T')} = e^{-c_2 m \left\{ \frac{1}{\lambda_c(T_s T_s')} - \frac{1}{\lambda_c(T T')} \right\}}$$

The value of R/R_0 is the correction factor by which the observed ratio R_0 of the brightness of the window must be multiplied in order to obtain the correct ratio R .

As an example we may take the case of the comparison of a tungsten lamp of colour-temperature 2400° K. with a standard carbon lamp of colour-temperature 2000° K. in an integrator whose selectivity is such as to lower the colour-temperature of a lamp from 2400° to 2250° .

Here

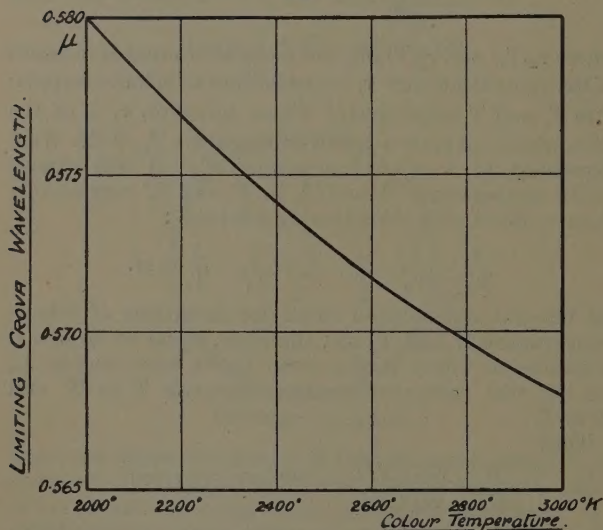
$$\begin{aligned} T_0 &= 2400^{\circ}, & T_0' &= 2250^{\circ}, & m &= 0.00002778; \\ T &= 2400^{\circ}, & T' &= 2250^{\circ}; \\ T_s &= 2000^{\circ}, & T_s' &= 1895^{\circ}. \end{aligned}$$

Since the mean Crova wave-length for the colour-temperature interval T to T' is the mean of the limiting Crova wave-lengths for colour-temperatures T and T' , we have from fig. 2, which shows the variation of the limiting Crova wave-length ⁽⁵⁾ with colour-temperature,

$$\lambda_c(TT') = 0.5751 \mu,$$

$$\lambda_c(T_s T_s') = 0.5809 \mu$$

Fig. 2.



Variation of limiting Crova wave-length
with colour-temperature.

and

$$\begin{aligned} \log_e \frac{R}{R_0} &= -c_2 \left(\frac{1}{0.5809} - \frac{1}{0.5751} \right) 0.00002778 \\ &= 0.00694, \end{aligned}$$

i. e.,

$$\frac{R}{R_0} = 0.993.$$

Thus the error due to the selectivity is 0.7 per cent.,

which agrees with the result obtained by Walsh and Barnett quoted earlier.

*Luminous Output Measurements on Luminous
Discharge-tubes.*

When sources such as the modern luminous discharge-tubes which emit more or less monochromatic light are to be measured in terms of standard gas-filled lamps of known colour-temperature, the difficulties of heterochromatic photometry make it desirable to modify the procedure. This can be done in two ways by means of colour-filters, which, when used in combination with a gas-filled lamp, modify the light emitted, so that it colour-matches that from the luminous discharge-tubes. The colour-match does not imply identity of energy distribution in the spectra of the two sources. The colour-match filter merely reduces the psychological difficulty of making a photometric measurement in the presence of considerable energy-distribution differences.

In the first method the colour-match filter is used in conjunction with the comparison lamp while measurements on the discharge-tube are being made. It is removed during the measurements on the gas-filled lumen standard lamp which takes the place of the discharge-tube in the integrator. The colour-temperature of the comparison lamp should in this comparison approximate to that of the lumen standard. The effect of the colour-filter is allowed for by using its integral transmission ratio computed for radiation having the colour-temperature of the comparison lamp.

In the second method an additional similar colour-match filter is used in combination with the lumen standards (in front of the window of the integrator). Its effect is allowed for by using its integral transmission ratio computed for radiation having the colour-temperature of the lumen standard.

There is very little difference in practice between the two methods. The second is probably preferable, as it is a better substitution method, the comparison lamp side of the photometer remaining the same throughout the measurements. In the very special cases when the colour-match filter gives an energy match as well, the second method gives perfect compensation for selectivity errors, whereas the first method does not.

*The Effect of Selectivity in the Case
of Single-lined Spectra.*

If the test source has a spectrum consisting of a single line, the brightness of the window when the test source is in the integrator is given for the first method by

$$B = pF(\lambda_1) \frac{r_{\lambda_1} t_{\lambda_1}}{1 - r_{\lambda_1}} K_{\lambda_1},$$

where $F(\lambda_1)$ is the energy emitted of wave-length λ_1 . Replacing the properties of the paint and window by the equivalent filter τ_λ , we have

$$R_0 = \frac{B}{B_s} = \frac{\tau_{\lambda_1} F(\lambda_1) K_{\lambda_1}}{a_s \int \tau_\lambda E(\lambda T_s) K_\lambda d\lambda}.$$

If the equivalent filter is non-selective, we have

$$R = \frac{F(\lambda_1) K_{\lambda_1}}{a_s \int E(\lambda T_s) K_\lambda d\lambda} = \frac{L}{L_s}$$

and

$$\begin{aligned} \frac{R}{R_0} &= \frac{\int \tau_\lambda E(\lambda T_s) K_\lambda d\lambda}{\int E(\lambda T_s) K_\lambda d\lambda} \cdot \frac{1}{\tau_{\lambda_1}}, \\ &= \frac{\tau_{\lambda_1} (T_s T_s')}{\tau_{\lambda_1}} = e^{-c_2 m \left\{ \frac{1}{\lambda_c (T_s T_s')} - \frac{1}{\lambda_1} \right\}} \end{aligned}$$

The curve in fig. 3 shows the variation of R/R_0 with wave-length, when the lumen standard has a colour-temperature of 2800° , for two amounts of selectivity, viz.,

(a) $m = 0.00000886$, i. e., 2400° K. to 2350° K.

(b) $m = 0.00002778$, i. e., 2400° K. to 2250° K.

Thus measurements on a sodium tube emitting light of wave-length 0.5890μ when compared with gas-filled standards of colour-temperature 2800° will be in error by 0.6 per cent. and 2 per cent. for the two amounts of selectivity respectively.

Radiations of wave-length 0.5700μ and 0.5707μ will be measured without error in the two cases, since these are the mean Crova wave-lengths for the colour-temperature intervals defined by the colour-temperature of the standard itself and its colour-temperatures as modified by the

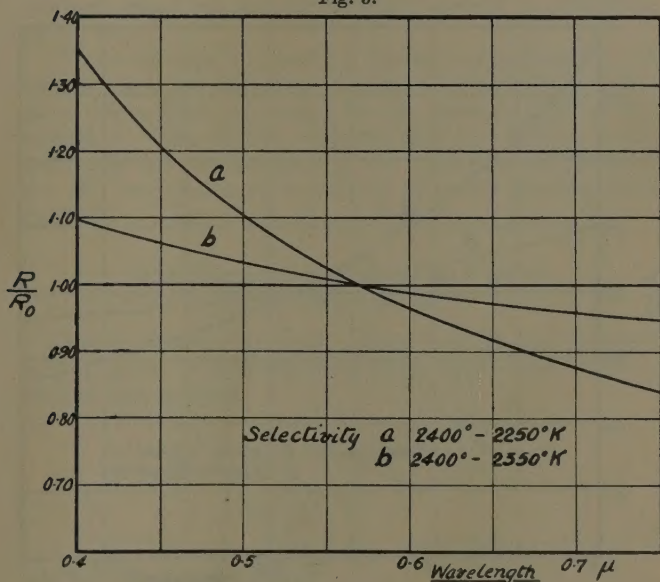
selectivities of the integrator. For radiation of wavelength 0.45μ the errors amount to 6 per cent. and 20 per cent.

For the second method we have

$$R_0 = \frac{\tau_{\lambda_1} F(\lambda_1) K}{a_s \int t_{\lambda} \tau_{\lambda} E(\lambda T_s) K_{\lambda} d\lambda},$$

$$R = \frac{F(\lambda_1) K_1}{a_s \int t_{\lambda} E(\lambda T_s) K_{\lambda} d\lambda} = \frac{L}{\mathcal{J} L_s},$$

Fig. 3.



Variation of correction factor with wave-length.

where t_{λ} is the transmission ratio of the colour-match filter used in conjunction with the lumen standard, and $\mathcal{J}$ is the integral transmission ratio of the colour-match filter for radiation having the colour-temperature of the lumen standard. Hence

$$\frac{R}{R_0} = \frac{\int t_{\lambda} \tau_{\lambda} E(\lambda T_s) K_{\lambda} d\lambda}{\int t_{\lambda} E(\lambda T_s) K_{\lambda} d\lambda} \cdot \frac{1}{\tau_{\lambda_1}},$$

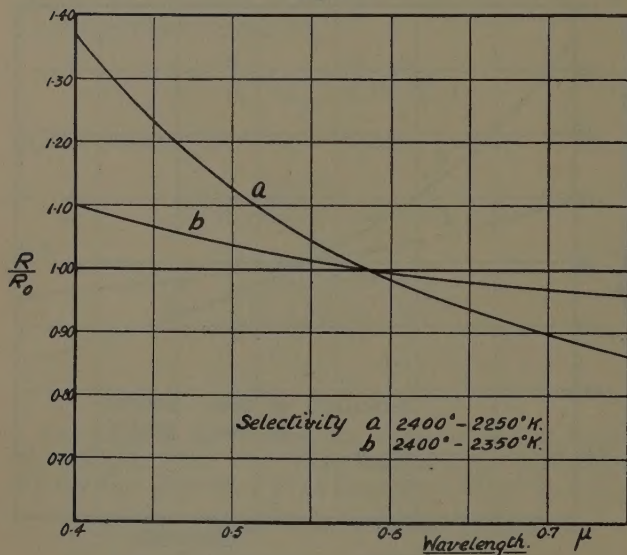
$$= \frac{\tau_{\lambda_s} (T_s T_s')}{\tau_{\lambda_1}} = e^{-c_2 m \left\{ \frac{1}{\lambda_s (T_s T_s')} - \frac{1}{\lambda_1} \right\}},$$

where λ_e is the mean effective wave-length of the colour-filter t_λ for the colour-temperature interval T_s to T_s' .

This follows since

$$\frac{\int t_\lambda \tau_\lambda E(\lambda T_s) K_\lambda d\lambda}{\int t_\lambda E(\lambda T_s) K_\lambda d\lambda} = \frac{k \int t_\lambda E(\lambda T_s') K_\lambda d\lambda}{\int t_\lambda E(\lambda T_s) K_\lambda d\lambda} \\ = k \cdot \frac{E(\lambda_e T_s')}{E(\lambda_e T_s)}$$

Fig. 4.



Variation of correction factor with wave-length for sodium lamp colour-match filter.

by the definition of mean effective wave-length ⁽⁶⁾. But

$$k \cdot \frac{E(\lambda_e T_s')}{E(\lambda_e T_s)} = k e^{-\frac{c_2}{\lambda_e} \left(\frac{1}{T_s'} - \frac{1}{T_s} \right)} \\ = k e^{-\frac{c_2}{\lambda_e} \left(\frac{1}{T_0} - \frac{1}{T_s} \right)} = \tau_{\lambda_e}.$$

The curves in fig. 4 show for the same amounts of selectivity as in the previous case the variation of R/R_0

with wave-length when the filter CF 51 whose spectral transmission curve is shown in fig. 6 is used in combination with a lumen standard of colour-temperature 2800° K. This combination gives a satisfactory colour-match with the light from a sodium vapour lamp.

It will be seen that the errors in measuring a sodium tube emitting radiation of wave-length 0.5890μ are less than 0.2 per cent. for both amounts of selectivity. These errors are considerably smaller than the corresponding errors deduced for the first method. Thus the second method, which is a true substitution method, is preferable to the first.

The errors would be zero by this second method for the measurement of radiation of wave-length 0.586μ if the same colour-filter were used for the measurement of this radiation. This follows from the fact that 0.586μ is the mean effective wave-length of the colour-match filter for not too large colour-temperature intervals starting from 2800° K.

*The Effect of Selectivity in the Case of
Multi-lined Spectra.*

Similar analysis as that carried out in the previous section gives, with sources emitting multi-lined spectra, for the first method

$$\begin{aligned} \frac{R}{R_0} &= \tau_{\lambda_c}(T_s T_s') \cdot \frac{\sum F(\lambda_n) K_{\lambda_n}}{\sum \tau_{\lambda_n} F(\lambda_n) K_{\lambda_n}} \\ &= \frac{\tau_{\lambda_c}(T_s T_s')}{\tau_{\Sigma}}, \end{aligned}$$

and for the second method

$$\frac{R}{R_0} = \frac{\tau_{\lambda_c}(T_s T_s')}{\tau_{\Sigma}},$$

where τ_{Σ} is the integral transmission of the effective filter τ_{λ} for the light emitted by the test-source.

The measurements of Krefft and Pirani ⁽⁷⁾ show that for the high-pressure mercury-vapour lamp (pressure = 800 mm.) more than 99 per cent. of the luminosity is due to radiations of wave-length 0.5780μ (0.5770μ and 0.5790μ) and 0.5461μ . For the low-pressure lamp the corresponding value is 98 per cent.

The relative luminosities of the three principal lines in the spectra are given below :—

λ .	Mercury-vapour lamp.	
	High pressure.	Low. pressure.
0.5770/90.....	52.3	23.3
0.5461.....	47.0	75.3
0.4358.....	0.7	1.4
	100.0	100.0

Calculations of the value of R/R_0 for the first method (colour-match filter on comparison lamp side only) gave the following values for the case when the selectivity is given by $m=0.00002778$ (*i. e.*, 2400° to 2250°) and a lumen standard lamp of colour-temperature 2800° K. is used :

High-pressure mercury lamp.... $R/R_0=1.012$

Low-pressure mercury lamp $R/R_0=1.024$.

These values are also obtained from fig. 3 by taking the value of R/R_0 for each of the three lines 0.5780μ , 0.5461μ , and 0.4368μ and forming their sum weighted for each line in the proportion of the luminosity of that line to the sum of the luminosities of the three lines.

Thus for the high-pressure lamp

$$\frac{R}{R_0} = \frac{523}{1000} \times 0.99 + \frac{470}{1000} \times 1.03 + \frac{7}{1000} \times 1.24 = 1.010,$$

and for the low-pressure lamp

$$\frac{R}{R_0} = \frac{233}{1000} \times 0.99 + \frac{753}{1000} \times 1.03 + \frac{14}{1000} \times 1.24 = 1.024.$$

Calculations for the second method (colour-match filter on each side when lumen standard is in the integrator) gave the following values in the same circumstances as those mentioned previously :

High-pressure mercury lamp.... $R/R_0=0.985$.

Low-pressure mercury lamp $R/R_0=0.992$.

These values are also obtained from fig. 5 as shown previously.

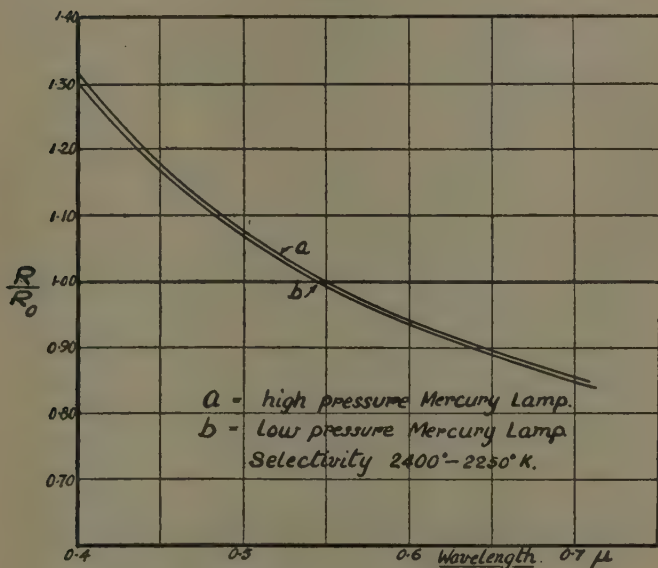
For the high-pressure lamp,

$$\frac{R}{R_0} = \frac{523}{1000} \times 0.965 + \frac{470}{1000} \times 1.003 + \frac{7}{1000} \times 1.21 = 0.985.$$

For the low-pressure lamp,

$$\frac{R}{R_0} = \frac{233}{1000} \times 0.957 + \frac{753}{1000} \times 0.998 + \frac{14}{1000} \times 1.20 = 0.991.$$

Fig. 5.



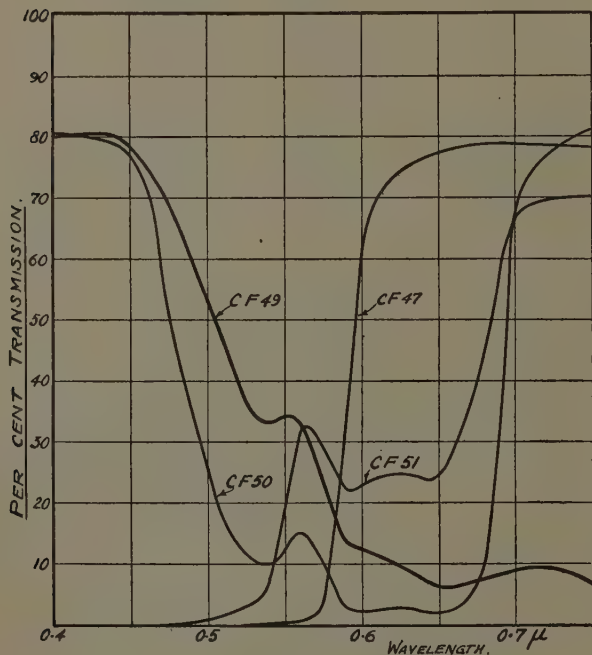
Variation of correction factor with wave-length for mercury lamp colour-match filters.

Fig. 5 shows the variation of R/R_0 with wave-length when the colour-match filter for high-pressure mercury lamps is used on the integrator side. The spectrophotometric transmission curve of the filter is shown as CF 49 in fig. 6. The spectrophotometric transmission curve of the colour-match filter used for low-pressure mercury vapour lamps is shown as CF 50 in the same figure. The mean effective wave-lengths of these filters for the colour-temperature interval 2800° to 2598° are 0.549 μ ,

and 0.544μ respectively. Thus the curve showing the variation of R/R_0 with wave-length for CF 50 is practically identical with that for CF 49 in fig. 5.

The effects for the two mercury lamps in each method are not very different in spite of a considerable difference in the distribution of energy in the two lines 0.5780μ

Fig. 6.



Spectro-photometric transmission curves of colour-match glasses used in the photometry of gaseous discharge lamps.

and 0.5461μ . This follows from the fact that the Crova wave-length of the lumen standard and the mean effective wave-length of the colour-match filter are both between these wave-lengths. Thus, on radiation of wave-length 0.5780μ the error produced by the selectivity is opposite in sign from that produced on radiation of wave-length 0.5461μ , so that the errors tend to cancel out.

Similar calculation for the case of the neon hot cathode lamp based on the energy measurements of Krefft and Pirani showed that for the same amount of selectivity and a lumen standard operating at a colour-temperature of 2800° K. the values of R/R_0 are as follows :—

First method $R/R_0 = 0.936$

Second method .. $R/R_0 = 0.990$.

As was found for the sodium and mercury low-pressure lamps, the effect of selectivity is less for the second method. The spectrophotometric transmission curve of the filter CF 47 which, with a tungsten lamp, colour-matches the neon light, is shown in fig. 6. Its mean effective wave-length for the colour-temperature interval 2800° to 2600° is 0.617μ .

Experimental Determination of the Effect of Selectivity.

Before the theory given previously had been worked out, an experimental determination of the effect of selectivity in the case of the high-pressure mercury lamp was made. This was done by imparting to the existing selectivity of the integrator additional selectivity by means of two Wratten filters, 86 C and 78 C, placed successively in front of the window of the integrator. The 86 C yellow filter increased the existing selectivity by about 160° and the 78 C blue filter reduced it by about 120° . The comparison of the results with those obtained with no additional filter at the window gave the following values of R/R_0 when the colour-temperature of the lumen standard was about 2800° K. :—

Method.	R/R_0 .			
	86 C.		78 C.	
	Observed.	Calculated.	Observed.	Calculated.
First method ..	1.014	1.012	0.985	0.99
Second method.	1.00	0.99	1.03	1.01

The agreement with theory is better than might be expected, as although colour differences were small,

energy distribution differences in the photometric comparisons were very great. In addition, the transmission curves of the filters 86 C and 78 C are not accurately of

the form $ke^{-\frac{c_2 m}{\lambda}}$, on which the theory is based. The results, however, established that quite considerable selectivity did not produce any serious error in the photometry of the high-pressure mercury lamp.

Conclusions.

The results given above show that for both methods, except in the case of the neon lamp, the error caused by a selectivity which produces a change of colour-temperature from 2400° to 2250° is not serious. If, however, the second method is used, in which a colour-filter is employed on the integrator side when the lumen standard is in the integrator, the errors become less than 2 per cent. in all cases.

The selectivity mentioned above is quite considerable. In general the selectivity of an integrator should be less than this amount, so that for the four types of discharge lamps considered the errors produced by selectivity should be negligible if the second true substitution method is adopted.

The calculations are based on the assumption that the selectivity is accurately of the type represented by the

expression $ke^{-\frac{c_2 m}{\lambda}}$. This need not necessarily be the case. The colour-match from which this expression is deduced need only be a sensation colour-match, and not an identity of energy distribution. For white paints, however, having zinc oxide, magnesium oxide, or carbonate as their basis, one would expect that even if the selectivity is not accurately defined by this expression the deviations from it would be of the second order over that part of the spectrum which contributes significantly to the total luminosity.

The author is indebted to Mr. F. J. C. Brookes and Mr. F. M. Hale, B.Sc., for the calculations of Crova wave-lengths and mean effective wave-lengths, and to Mr. W. Barnett and Mr. F. M. Hale for the experimental determination of the effect of additional selectivity occasioned by the use of the Wratten filters.

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LVIII. *X-ray Study of Aluminium-Zinc Alloys at Elevated Temperatures.* By Prof. E. A. OWEN and LLEWELYN PICKUP, M.Sc. (Lond.), Ph.D. (Wales), University College of North Wales, Bangor*.

THE most important equilibrium diagrams of the aluminium-zinc system which have hitherto been put forward are reproduced in a paper by Isihara †. Of these, the diagrams of Hanson and Gayler ‡, Tanabe §, and Isihara are the most recent, and, in general, they show similar features. Hanson and Gayler adopted the usual method of quenching, followed by microscopical examination, while Tanabe and Isihara employed differential heating and cooling curves and resistance-temperature measurements. Apart from establishing the first complete diagram, and especially the existence and the boundary of the β -phase region, Hanson and Gayler showed clearly the spontaneous decomposition of the β -phase at about 270°C ., a fact which undoubtedly complicated the difficulties of previous investigators when interpreting microstructures. The rapidity of this decomposition may be realized from one of their observations; they say that alloys with about 20 per cent. aluminium, after quenching from 360°C . in water, decompose into $(\alpha + \gamma)$ in $3\frac{1}{2}$ minutes, with an increase in temperature of about 50 to 60°C . The methods adopted by Tanabe and Isihara overcome such a difficulty, but to obtain data representing the true equilibrium state with their methods it is necessary for the constitution of the alloy to keep in step with the

* Communicated by the Authors.

† Isihara, J. Inst. Metals, xxxiii. p. 73 (1925).

‡ Hanson and Gayler, J. Inst. Metals, xxvii. p. 267 (1922).

§ Tanabe, J. Inst. Metals, xxxii. p. 415 (1924).

changing temperature. The rapid decomposition of the β -phase suggests that this condition is likely to obtain.

The present paper deals with an X-ray investigation of this system, and is a continuation of previous work done on the system in this laboratory *. With X-ray analysis the phases in duplex regions can be studied independently, and accurate measurements of parameter values can be taken. Then, by methods, of extrapolation and interpolation, unique points, such as phase boundaries and transformation temperatures †, can be obtained with precision.

To cope with such difficulties as the spontaneous decomposition encountered in this alloy system, a special X-ray technique has been developed, in order to obtain photographs of the samples while at elevated temperatures *in vacuo*.

At the time when the results of the previous X-ray investigation carried out on the system in this laboratory were published, only the work of Phebus and Blake ‡ was mentioned. Later it was found that another X-ray investigation of the system had been carried out by Schmidt and Wassermann §. Our results agreed with theirs in reference to the structures of the β - and γ -phases.

Discussing the constitution of a phase such as the β , lying between the simple solid solutions, α and γ , Hanson and Gayler state that "such a phase must possess a space lattice which is essentially different from that of its neighbours in the equilibrium diagram." Experiment shows that the β - and γ -phases are identical in structure, both being face-centred cubic, and it is further shown in the present work that these phases have, at about 360° to 400° C., almost the same parameter values. These facts render quenching and microscopical methods of investigation difficult. That the difficulties were successfully overcome by Hanson and Gayler is evident from the fact that their results have been confirmed in general by later workers who employed different methods of investigation. The methods adopted by Tanabe and Ishihara do not involve quenching, but in view of the marked

* Owen and Iball, *Phil. Mag.* xvii. p. 433 (1934).

† Owen and Pickup, *Proc. Roy. Soc. A*, cxxxvii. p. 397 (1932); cxlv. p. 258 (1934).

‡ Phebus and Blake, *Phys. Rev.* xxv. p. 107 (1925).

§ Schmidt and Wassermann, *Metallwirtschaft*, xi. p. 386 (1932).

differences in some of their data, and especially since their methods were the same in principle, it was considered desirable to obtain independent data by another method.

Apparatus and Experimental Method.

The most probable sources of error in the X-ray investigation of Owen and Iball were the measurement of temperature, the possible volatilization of zinc at high temperatures, and the use of an atmosphere of nitrogen. With the object of overcoming these difficulties, the authors have designed a new apparatus with which accurate measurement of parameter values can be obtained from samples heated *in vacuo*. The camera was of the Debye-Scherrer type and gave rotation powder photographs. The time of exposure was reduced to 30 minutes ; this was made possible by the slit system used and by employing an X-ray tube furnished with an incandescent cathode. With such a short exposure no measurable amount of volatilization took place. Details of the design and construction of the apparatus will be published later. The powder photographs gave the crystal structures ; also by adopting the extrapolation method developed by Bradley and Jay * they yielded parameter values of high accuracy.

Fine filings of each alloy were mounted on a silica rod, and this coated rod was surrounded with an electrical heating unit, which allowed the entry and exit of the X-ray beam issuing from the slit system and which did not obstruct the reflexions from the powder.

The sample was heated and photographed in a vacuum (about 0.001 mm. mercury) maintained in a brass enclosure which contained the camera, the heating unit, the mechanism to rotate the sample, and water-jackets to keep the film cool. The general procedure was first to evacuate the enclosure for half an hour, then to pass an accurately known current through the heater from a large storage battery for sufficient time to bring about equilibrium in the sample, and, finally, to expose the sample to the X-ray beam, which entered the enclosure through a cellophane window. Throughout this work radiations from cobalt and chromium anticathodes were used, as the range of parameter values to be measured

* Bradley and Jay, Proc. Phys. Soc. xliv. p. 563 (1932).

was such that suitable reflexion lines with these particular wave-lengths were registered at large glancing angles—a condition for the greatest accuracy.

Direct application of any of the usual forms of pyrometer to measure accurately the temperature of so small a sample was not found possible. Temperature measurements were obtained from photographs of pure silver heated *in vacuo* with different but accurately known currents passing through the heating unit. The parameter of silver at different temperatures was found from the X-ray photographs; then, using the thermal expansion curve of pure silver, which has been well established by different methods, the temperature produced in the sample by a known current was obtained. In this way the relation between temperature of the irradiated sample and the current passing through the heater was found up to 400° C. Consistent temperature measurements could be repeated to within $\pm 3^\circ$ C., by maintaining standardized conditions.

Owen and Iball had found that a preliminary annealing of these alloys in the ingot form to produce a homogeneous composition was not essential. Most of their alloys, some of which had been lump annealed, were used in the present work. The use of filings in X-ray analysis has been criticized on the grounds of possible variations in composition from particle to particle. The object of annealing in lump form is, of course, to avoid such heterogeneity in the sample; the experimental evidence in the present work showed that the samples used were homogeneous. To obtain the time of annealing required for thermal equilibrium, samples were heated *in vacuo* for different times. After half an hour's heating the appropriate structures were obtained above and below the eutectoid temperature, while after 6 hours' heating the measurements and the appearance of the reflexion lines were identical with those after half an hour's heating. That the filings have distorted structures before annealing undoubtedly hastened the attainment of equilibrium. A more important point, however, regarding the use of filings of aluminium-zinc alloys is the possibility of the volatilization of the zinc at elevated temperatures. Work* on copper-zinc alloys at temperatures up to 400° C. in air showed that no measurable amount of

* Owen and Pickup, *loc. cit.*

volatilization took place in the short interval during which the sample was kept at high temperature. Samples in the present work when photographed before and after heating a number of times at temperatures ranging between 360° C. and 400° C. gave identical results, and the appearance of the reflexion lines remained unaltered. It was concluded that the composition could only be negligibly changed, if at all, and, therefore, any volatilization which may have taken place up to 400° C. during the experiment did not impair the results from the different alloys.

Experimental Results.

The chief features of the hitherto published equilibrium diagrams which required re-determination were the extent of the γ -phase region above 270° C. and the extent of the β -region, features which are important in connexion with the age-hardening properties of this alloy system as discussed by Hanson and Gayler. It was also necessary to re-check the eutectoid temperature, especially as Owen and Iball reported this to be higher than that given by previous investigators.

Ingots weighing about 30 grams, and of compositions extending over a wide range, were prepared from commercially pure zinc and aluminium*, by melting together weighed quantities. As a check on the composition, some of these ingots were chemically analysed. All the alloys in the β -phase region (18 to 30 per cent. aluminium) were analysed for zinc. The results of the chemical analyses are given in Table I.

When determining phase-boundaries from photographs taken at elevated temperatures, the procedure is the same as that for quenched samples, that is, one finds within the duplex region that the parameter value of each phase at a constant temperature is the same and independent of composition. It usually changes with composition in the pure regions. In photographs of quenched samples taken at room-temperatures the only effect determined from parameter measurements is any change of solubility with temperature†. In the work at elevated temperatures

* Ishihara found that the effect of impurities (0.2 per cent. Si, 0.3 per cent. Fe) is negligibly small by comparison with alloys made from specially pure metals.

† We assume here that the condition and the constitution of an alloy at the quenching temperature is retained, except for thermal contraction.

it is necessary to determine the relation between parameter value and composition in the pure and mixed regions for each temperature, as, in addition to a solubility effect, there is also an effect due to thermal expansion.

TABLE I.

Alloy No.	Zn.	Al.	Cu.	Fe.
85 A....	15.12	84.15	0.21	0.51
60.....	38.23	61.34	—	—
50.....	50.88	48.07	0.29	0.65
40.....	59.15	40.8*		
33.....	69.53	30.5*		
30.....	71.10	28.9*		
28.5....	73.31	26.7*		
25.....	76.4	23.6*		
23.....	80.52	19.5*		
21.....	81.69	18.3*		

* By difference.

TABLE II.

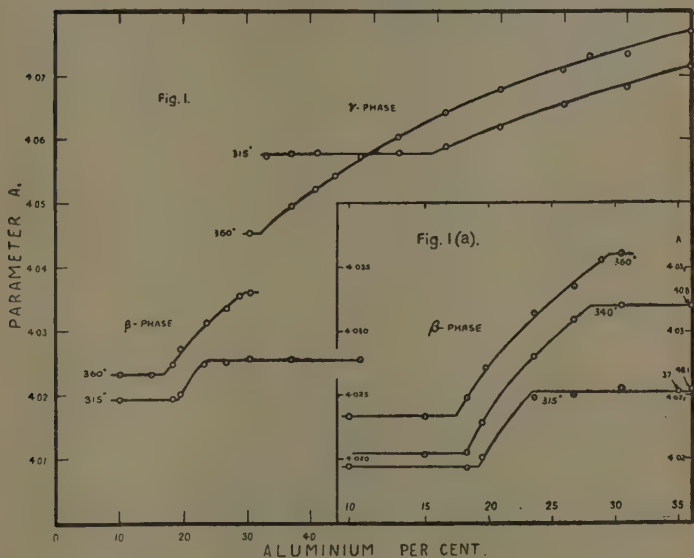
Parameter Values at 360° C.

γ -phase.		β -phase.	
Al (wt. per cent.).	Parameter (Å.).	Al (wt. per cent.).	Parameter (Å.).
100	4.0770	30.5	4.0360
90	4.0735	28.9	4.0355
84.2	4.0730	26.7	4.0335
80	4.0708	23.6	4.0314
70	4.0678	19.5	4.0271
61.3	4.0640	18.3	4.0248
54	4.0602	15	4.0233
48.1	4.0574	10	4.0233
44	4.0540		
40.8	4.0520		
37	4.0495		
30.5	4.0448		

A series of alloys was taken, representing compositions across the whole diagram, and photographed at 360° C. *in vacuo*, with cobalt and chromium radiations. The reflexions measured were from the (331) and (420) planes with cobalt, and the (311) and (222) planes with chromium. The results are given in Table II.

In the range of composition, 25 to 35 per cent. aluminium, well-defined reflexion lines at the high glancing angles were sometimes difficult to obtain. This was attributed to the effect of temperature on the intensity of the reflexions and to the large increase in the γ -phase parameter with fall of temperature, as will be shown later. Slight changes or fluctuations in temperature would, therefore, cause the reflexion lines to broaden and become

Figs. 1 & 1 (a).



diffuse. Many photographs were taken in this region. The data in Table II. are shown graphically in fig. 1. In addition to showing that the structures of the β - and the γ -phases are the same (face-centred cubic), the photographs yield data (fig. 1) which show that the parameter values of alloys at 360° C. in these phases form almost a continuous relation with composition. A few photographs of alloys taken at 400° C. indicated that this relation is still more strictly continuous at this temperature.

A second series of photographs was taken at 315° C.

The results are contained in Table III. and are shown in fig. 1.

A third series taken at 340° C. with alloys containing 15 to 40 per cent. aluminium, and representing the β -phase

TABLE III.
Parameter Values at 315° C.

γ -phase.		β -phase.	
Al (wt. per cent.).	Parameter (Å.).	Al (wt. per cent.).	Parameter (Å.).
100	4.0717	48.1	4.0254
90	4.0680	37	4.0253
80	4.0653	30.5	4.0255
70	4.0615	26.7	4.0250
61.3	4.0588	23.6	4.0248
54	4.0579	19.5	4.0201
48.1	4.0576	18.3	4.0193
40.8	4.0579	10	4.0194
37	4.0577		
30.5	4.0573		

TABLE IV.
Parameter Values at 340° C.

β -phase.	
Al (wt. per cent.).	Parameter (Å.).
40.8	4.0320
30.5	4.0322
26.7	4.0309
23.6	4.0280
19.5	4.0229
18.3	4.0205
15	4.0203

only, are given in Table IV. These results are shown on an enlarged scale in fig. 1 (inset, 1 (a)), with other results concerning the β -phase contained in Tables II., III., and IV. From these data, the compositions of the boundaries at 315°, 340°, and 360° C. are found; these are shown in Table V.

The Eutectoid Transformation $\beta \rightarrow (\alpha + \gamma)$.

To determine the temperature of the eutectoid transformation, photographs of certain alloys were taken over a range of temperatures extending on either side of 270°C . Since the crystal structures of the different phases were known, photographs of alloys containing $(\beta + \gamma)$ above and $(\alpha + \gamma)$ below the transformation temperature were examined to ascertain at what temperature the structure of the β -phase was replaced by the α phase structure. Only an approximate estimate could be made in this way. The expansion curve of the γ -phase in alloys containing 30.5, 40.9, and 54.0 per cent. aluminium gave accurately, in a somewhat unexpected manner, both the eutectoid transformation temperature and the temperature at

TABLE V.
Aluminium (per cent. by weight).

Boundary.	360° C.	340° C.	315° C.
$(\alpha + \beta) - (\beta)$	17.4	18.0	19.0
$(\beta) - (\beta + \gamma)$	29.5	28.0	23.5
$(\beta + \gamma) - (\gamma)$	32.0	..	60.4

which each of these compositions is on the $(\beta + \gamma) - (\gamma)$ boundary. The results given by these alloys are contained in Table VI. and are illustrated in fig. 2. Since the γ -phase parameter of these alloys in the $(\alpha + \gamma)$ and $(\beta + \gamma)$ regions at the same temperature are the same, the curves representing the 40.9 and 54.0 alloys have for clearness been displaced relative to that of the 30.5 alloy along the parameter axis.

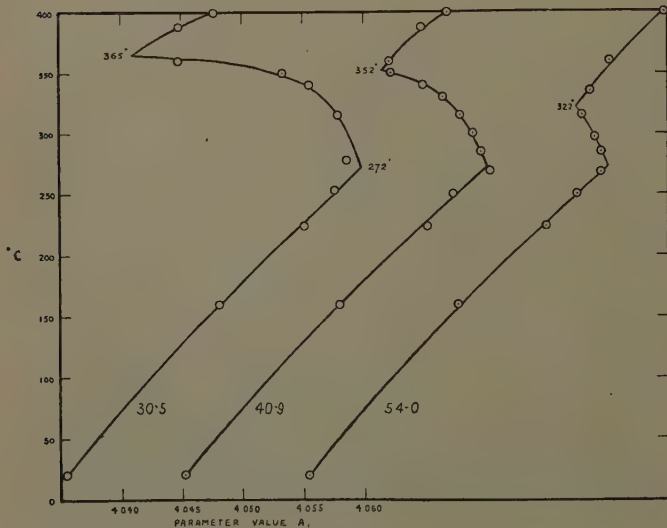
That the curves in fig. 2 give such definite information on the eutectoid temperature is due to the rapid increase of the solubility of zinc in the γ phase lattice with increasing temperature above the transformation temperature. Fig. 1 has already shown that, as the zinc content of the γ -phase increases, the parameter value becomes smaller. Fig. 2 shows further that this decrease of parameter value with increasing solubility of zinc in the lattice, as the temperature increases, is greater than the thermal expansion of the γ -phase lattice over the same range of temperature within the $(\beta + \gamma)$ region. This "negative"

TABLE VI.

 γ -phase Parameter at Different Temperatures.

Alloy 30.5.		Alloy 40.9.		Alloy 54.	
Temp. (°C.).	Parameter (Å.).	Temp. (°C.).	Parameter (Å.).	Temp. (°C.).	Parameter (Å.).
20	4.0355	20	4.0352	20	4.0354
160	4.0481	160	4.0480	160	4.0471
224	4.0551	224	4.0552	224	4.0550
253	4.0576	250	4.0573	250	4.0575
269	4.0586	269	4.0604	268	4.0595
315	4.0570	285	4.0596	285	4.0595
340	4.0555	300	4.0580	297	4.0590
350	4.0533	315	4.0579	315	4.0579
360	4.0448	330	4.0565	335	4.0586
365	4.0410	340	4.0549	360	4.0602
388	4.0448	350	4.0522	400	4.0647
400	4.0477	360	4.0521		
		388	4.0548		
		400	4.0568		

Fig. 2.



expansion of the γ -phase in the $(\beta+\gamma)$ region is interesting, as, in so far as the authors are aware, such a phenomenon has not been previously reported. In the range of temperature below the eutectoid temperature there is little change in the solubility of zinc in the γ -phase lattice, and hence the slope of the γ -phase curve is positive. At the higher temperatures, where the composition of an alloy passes from the $(\beta+\gamma)$ to the unsaturated γ -region, only the thermal expansion can change the parameter, and so a normal thermal expansion curve is obtained. It is clear, therefore, that, owing to peculiar features in this particular alloy system, X-ray data from "expansion" curves alone enable one to determine accurately (by

TABLE VII.
 β -phase Parameter Values.

Temp.	Alloy composition.			
	23.6	30.5	40.9	54
285....	4.0195	4.0197	4.0200	—
290....	4.0200	—	—	4.0195
297....	4.0212	—	4.0213	4.0210
315....	4.0242	4.0255	4.0249	4.0254
330....	—	—	4.0290	—
340....	4.0278	4.0320	4.0322	—
350....	—	4.0338	—	—
360....	4.0314	—	—	—
390....	4.0400	—	—	—

interpolation) both the eutectoid temperature and points on the $(\beta+\gamma)-(\gamma)$ boundary.

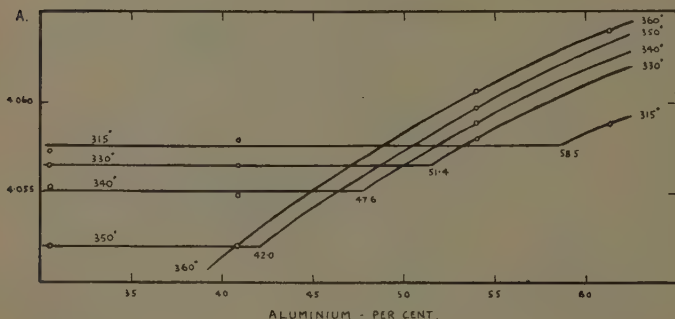
From the data shown in fig. 2 the eutectoid temperature is found to be at 272° C., compared with 256° C. by Hanson and Gayler, 270° C. by Tanabe, and 280° C. by Ishihara. The $(\beta+\gamma)-(\gamma)$ boundary at 30.5, 40.9, and 54.0 per cent. aluminium is found, from fig. 2, to be at 365°, 352°, and 322° C. respectively.

From the many photographs taken with these three alloys, and an alloy containing 23.6 per cent. aluminium, the parameter values of the β -phase were obtained (Table VII.). These values are only approximate, especially near the eutectoid temperature, because reflexion lines from the β -phase were weak and difficult to measure.

These results are sufficient to show, firstly, that the β -phase has in each alloy a positive expansion with temperature; secondly, that each alloy has the same parameter value up to about 315°C. ; and, thirdly, that the β -parameter in the alloy containing 23.6 per cent. aluminium does not expand as rapidly as that in the other three alloys above this temperature. This is consistent with the fact that this particular alloy passes at about 315°C. from the $(\beta+\gamma)$ into the pure β region, while the other alloys remain in the $(\beta+\gamma)$ region, where the increase in parameter value due to the change of solubility with temperature persists.

Further, by combining the data in Table VI. with those in Tables II. and III., the $(\beta+\gamma)-(\gamma)$ boundary at 330° ,

Fig. 3.



340° and 350°C. has been obtained in the following manner. Over the small range of temperature ($315-360^{\circ}\text{C.}$) the relation between composition and parameter at each of these temperatures in the pure region is assumed* to be parallel to those at 315 and 360°C. (determined in fig. 1). Fig. 3 shows a portion of the curve at 360°C. , with parallel curves drawn through the parameter values of alloy 54 at 330° , 340° , and 350°C. (taken from fig. 2); this alloy is in the pure region at these temperatures. The alloys containing 40.9 and 30.5 per cent. aluminium are in the $(\beta+\gamma)$ region at these temperatures, and the

* In effect, the coefficient of thermal expansion of the unsaturated γ -phase over a range of composition is assumed to be the same.

γ -phase parameter has a constant value for each temperature, as shown in Table VIII., which is compiled from fig. 2.

TABLE VIII.
Parameter (Å.).

Temp. °C.	Aluminium (per cent. by wt.).		
	54	40.9	30.5
360....	4.0606	4.0521	4.0448
350....	4.0597	4.0522	4.0520
340....	4.0589	4.0549	4.0552
330....	4.0581	4.0565	4.0565

From fig. 3 the $(\beta + \gamma) - (\gamma)$ boundary is found to be, at 51.4, 47.6, and 42.0 per cent. aluminium, at 330°, 340°, and 350° C. respectively.

Discussion of the Results.

The points on the phase-boundaries obtained in this investigation are summarized below :—

(a) From Parameter-composition Curves.

Boundary.	360° C.	350° C.	340° C.	330° C.	315° C.
	Al (per cent.).				
$(\alpha + \beta) - (\beta)$...	17.4	—	18.0	—	19.0
$(\beta) - (\beta + \gamma)$...	29.5	—	28.0	—	23.5
$(\beta + \gamma) - (\gamma)$...	32.0	42	47.6	51.4	58.5

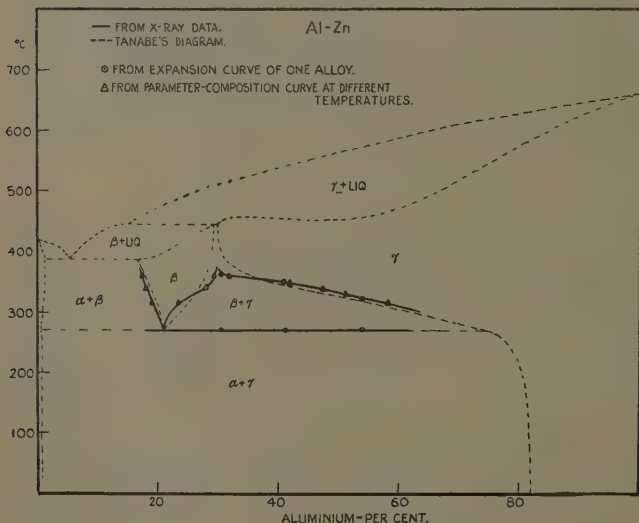
(b) From Expansion Curves.

Al (wt. per cent.).	Temp. of $(\beta + \gamma) - (\gamma)$ boundary.
30.5	365° C.
40.9	352° C.
54	322° C.

In order to give a graphical comparison, these results are shown in fig. 4, together with the diagram of Tanabe, with which they are in most general agreement. The pure β -phase by extrapolation is found to meet the

horizontal eutectoid line at about 21 per cent. aluminium, in agreement with Hanson and Gayler, Tanabe, and Isihara. The extent of the eutectoid line probably reaches to about 75 per cent. aluminium, as given by Tanabe. Hanson and Gayler, and Isihara give this point at about 65 per cent. Between this point and 33 per cent. aluminium the $(\beta+\gamma)-(\gamma)$ boundary up to 360°C . is in good agreement with that given by Tanabe. While

Fig. 4.



the $(\beta+\gamma)$ region above 360°C . given by Tanabe covers a more restricted range of composition than in Isihara's diagram, the present investigation suggests that this region is still more restricted. The decrease in parameter value of the γ -phase at about 400°C . with increasing zinc content is such that this parameter value approaches that of the β -phase in this region of temperature at about 30 per cent. aluminium. The present X-ray data do not confirm the existence of the $(\beta+\gamma)$ region above about 360°C ., but, if it exists, it covers a range of composition much less than 1 per cent.

The observations of Owen and Iball regarding the structures of these alloys below the eutectoid temperature have been confirmed. It appears difficult to account for the difference between their results and those obtained in the present investigation at elevated temperatures. With the present experimental arrangement much more accurate measurements of parameter values and temperatures were made. Without definite information about the effect of nitrogen on these alloys when heated, one does not know to what extent such an effect may have contributed to the divergence between the results of the two investigations. That this divergence is produced by volatilization of zinc does not appear tenable, since in both investigations there are good reasons to believe that no measurable volatilization had taken place, at least up to 400°C . An investigation* into the interdiffusion of copper and zinc suggests that the amount of volatilization of zinc which takes place below its melting-point, 417°C ., in these alloys has no appreciable effect on the parameter values.

The eutectoid transformation is found to be, at 272°C ., in good agreement with the value by Tanabe. It is interesting to note that Isihara, using a differential dilatometer, finds this temperature to be 280°C . Unlike the curves of the γ -phase used in this work, Isihara's curves represent the alloy in bulk, when an expansion over the whole range of temperature is found, with a marked increase occurring at the eutectoid temperature. The change in parameter value of the saturated β -phase in the $(\beta+\gamma)$ region due to the solubility effect is in the same direction as that due to the thermal effect, so that, to account for Isihara's expansion curves, this total increase in expansion of the β -phase with temperature must be greater than the resultant decrease in parameter of the γ -phase recorded in this investigation. Given sufficient time, these effects are reversed in the two phases when the temperature is lowered. A pertinent question is, do such circumstances have any relation to the age-hardening shown by these alloys, especially at temperatures where the relative amount of the β -phase is small? It may be recalled that Hanson and Gayler found age-hardening in alloys containing 40 to 80 per cent. alu-

* Owen and Pickup, Proc. Roy. Soc. A, cxlix. p. 282 (1935).

minium slight at 80 per cent., but appreciable at 75 and 70 per cent., while alloys containing 65 to 40 per cent. aluminium were not found to age-harden spontaneously at 0° C. within a period of three or four days. These investigators suggest that this difference in behaviour is due to the somewhat different constitution of the alloys.

Isihara finds that the peritectic line at 440° C. extends to 41 per cent. aluminium; the present results indicate that it extends to about 30 per cent., as given by Hanson and Gayler and by Tanabe. As pointed out by Gayler, one would not have expected such divergent results from electrical conductivity measurements as those given by Tanabe and by Isihara. Concerning this, the present results support the suggestion of Gayler that Isihara's measurements may not have been made on alloys in true equilibrium.

A new feature introduced by Isihara is the transformation in the $(\alpha+\beta)$, β , and $(\beta+\gamma)$ region at 340–350° C., which he described as a progressive change in the β -phase like the A2 transformation in iron. The authors* have previously investigated such a transformation by X-ray methods, but a careful investigation concerning thermal expansion and intensity measurements of the reflexion lines is necessary to throw any light on this suggestion. No conclusions on this point have been drawn from the present results.

Summary.

Apparatus has been designed to obtain rotation powder photographs of metals and alloys at elevated temperatures *in vacuo* and to obtain accurate measurements of parameter values and temperature. Such apparatus is essential for the X-ray study of the β -phase in aluminium-zinc alloys, owing to its spontaneous decomposition below 270° C., even after severe quenching. Apart from confirming the structure of the phases already given in other investigations, the present work mainly concerns the structure, parameter values, and phase-boundaries above the eutectoid temperature.

Of the recent equilibrium diagrams put forward, that of Tanabe is in the most general agreement with the present X-ray results. The only marked divergence is

* Owen and Pickup, Proc. Roy. Soc. A, cxlv. p. 258 (1934).

in the extent of the $(\beta+\gamma)$ region above about 360°C. ; in the present work it is found to extend over about 3 per cent. at 360°C. , compared to about 8 per cent. in Tanabe's diagram at this temperature. Above this temperature the $(\beta+\gamma)$ region up to the peritectic line at 440°C. is, therefore, found to occupy an exceedingly narrow range of composition.

Owing to the large decrease in parameter as zinc dissolves in the γ -phase above 270°C. , this phase shows a decrease in parameter as the temperature increases. As a result of this, both the transformation temperature and the $(\beta+\gamma)-(\gamma)$ boundary have been determined from a study of the thermal "expansion" curves of the γ -phase in certain alloys. The transformation temperature is found to be at 272°C. , compared to 270° , 280° , and 256°C. by Tanabe, Isihara, and Hanson and Gayler respectively.

The phase-boundaries obtained from this X-ray investigation are, in aluminium per cent. by weight:— $(\alpha+\beta)-(\beta)$: 17.4 (360°C.), 18.0 (340°C.), 19.0 (315°C.); $(\beta)-(\beta+\gamma)$: 29.5 (360°C.), 28.0 (340°C.), 23.5 (315°C.); $(\beta+\gamma)-(\gamma)$: 32.0 (360°C.), 42.0 (350°C.), 27.6 (340°C.), 51.4 (330°C.), 58.5 (315°C.), 30.5 (365°C.), 40.9 (352°C.), 54 (322°C.).

A new feature shown by the X-ray measurements is that the γ -phase parameter in the $(\beta+\gamma)$ region decreases with increasing temperature. It is suggested that this feature may produce the age-hardening found in alloys in the composition of which there is comparatively little β -phase.

We wish to record our indebtedness to the Department of Scientific and Industrial Research for the grant which enabled us to carry out the investigation.

9th July, 1935.

LIX. *Air Afterglow.* By F. H. NEWMAN, *Ph.D., D.Sc.*,
Professor of Physics, University College, Exeter *.

[Plate VI.]

THE character of the afterglow in a mixture of nitrogen and oxygen depends upon the total gas pressure, the percentage of oxygen present in the mixture, the

* Communicated by the Author.

Phil. Mag. S. 7. Vol. 20. No. 135. Nov. 1935. 3 E

presence of impurities such as water-vapour, etc., and the nature of the electrical discharge, *i. e.*, whether electrodeless or the ordinary type of electrode discharge *. In well baked-out vessels the gas mixtures subjected to an electrodeless discharge show little, if any, glow, and this also applies to pure oxygen and pure nitrogen. It seems that the effect is only produced when some poisoning substance is present on the walls of the discharge-tube. The nature of this "active" constituent is not definitely known—it may be water-vapour or possibly some high oxide of nitrogen. As shown by Lord Rayleigh †, the afterglow obtained when an electrode discharge is passed through air is of a chemiluminescence nature, due to the reaction of ozone upon nitric oxide—both gases being produced by the discharge through the oxygen-nitrogen mixture. The same type of glow is also formed by mixing these two gases, *viz.*, ozone and nitric oxide, the latter being prepared by one of the more usual chemical reactions. Rayleigh found that the afterglow spectrum was continuous, extending from λ 4200– λ 6700. Stoddart ‡, who has examined the spectrum of oxygen afterglow, noted that it was not strictly continuous, but consisted of 28 or 30 very weak diffuse bands spaced at equal intervals over the range λ 4200– λ 6700. On the whole air and oxygen afterglows have similar spectra as regards continuity over this range of wave-lengths.

Similar glows, although of different colours, are produced by the action of ozone upon other oxidizable vapours such as hydrogen sulphide, carbon bisulphide, acetylene and cyanogen §. In most of these glows the spectra are continuous, although in some cases various bands are present, together with the continuity. All such glows show the characteristics of flames, but their temperature is little above that of the surroundings.

In the present work the spectra of air and oxygen afterglows have been examined in more detail. A regulated current of gas was drawn through a vacuum tube by a powerful pump, and the afterglow examined in a side tube. While photographing the results a little stray

* See Lewis, *Journ. Amer. Chem. Soc.* li. p. 665 (1929), and Herzberg, *Phys. Zeits.* xlv. p. 878 (1928).

† Strutt, *Proc. Phys. Soc.* xxiii. p. 66 (1910).

‡ *Proc. Roy. Soc. A*, cxlvii. p. 454 (1934).

§ See Strutt, *loc. cit.*

light from the discharge-tube was liable to find its way by reflexion into the side tube, but the lines thus recorded were always faint, and could be eliminated by comparison with a direct photograph of the spectrum of the discharge. A constant deviation spectrograph was used and good photographs of the spectra of the afterglows were obtained on hypersensitive panchromatic plates with an hour's exposure. The intensity of the glow depended upon the total pressure of the gases in the discharge-tube, 0.2 mm. of mercury giving the maximum intensity. Exposures of eight hours with a 2-metre grating gave no indication of spectra. The discharge-tube was of pyrex, with tungsten seals and aluminium electrodes, the latter being well baked before and after setting up the apparatus. The gases used were air and cylinder oxygen (99.5 per cent. pure).

When the effect of vapours on the afterglows was to be investigated a small tube containing the appropriate liquid was attached to the tube in which the glow was observed. Such vapours were prevented from entering the pump by being passed over hot copper oxide in a tube.

With air it was found that if the discharge-tube were cleared and dried before passing the gas, the afterglow was at first very faint at all pressures, but after the discharge had been passing for some time the intensity increased. This increased intensity could also be brought about by introducing water-vapour into the discharge-tube. Once an intense glow had been produced it could always be immediately obtained in any subsequent discharge, however long the interval between the experiments. This points to the fact, as noted by other experimenters, that some poisoning of the discharge-tube walls is necessary for the glow formation, possibly by water-vapour, or some product formed during the discharge, *e. g.*, one of the higher oxides of nitrogen. This poisoning effect persists after the discharge has ceased, but is eliminated by heating the tube.

The colour of the afterglow varies slightly with the pressure. Below 0.2 mm. of mercury it is more greenish in colour and the spectrum extends further into the violet region than it does at higher pressures. The spectrum is continuous, extending from λ 4200 to λ 6700, but in addition to this continuity there is a series of diffuse bands—difficult to measure on account of their faintness

and the presence of the continuous spectrum. Microphotometer records, however, enable the positions of the bands to be approximately located, and they are as follows — λ 4130, 4190, 4250, 4260, 4300, 4340, 4385, 4410, 4440, 4470, 4500, 4525, 4625, 4705, 4730, 4800, 4840, 4950, 5020, 5050, 5150, 5230, 5370, 5460, 5550, 5680, 5980, 6320, and 6400—29 bands in all, but there are probably others masked by the continuous spectrum. Some of the bands can be seen towards the violet end of the spectrum in the spectrogram (Pl. VI.). The α -bands of nitrogen are also evident in the original negatives, indicating the presence of the ordinary nitrogen afterglow in the tube. Stoddart found that the oxygen afterglow continuous spectrum consisted of 28 or 30 very weak diffuse bands spaced at equal intervals over the range λ 4200— λ 6700. The present work indicates that the bands in air are superposed upon a continuous spectrum.

The air afterglow is quenched by water-vapour admitted from the side tube, *i. e.*, the water-vapour would be at saturation pressure : further, the glow did not reappear along the tube. Similar results, *i. e.*, extinction of the afterglow, were produced by the vapours of acetone and benzene. On the other hand, carbon bisulphide vapour gave a blue flame at the place where it entered the air-glow stream, and this blue glow extended along the tube, the extent depending upon the pressure of the carbon bisulphide vapour. At very small pressures of the latter the familiar yellow-green air glow reappeared beyond the blue glow. This is attributed to the fact that the amount of nitric oxide existing at the entrance of the carbon sulphide is more than sufficient to react with the latter, and thus the combination of the ozone and nitric oxide is resumed or is proceeding along the tube, but the glow thus produced is masked over a distance by the brilliant blue flame of the carbon bisulphide. Ozone also reacts with the latter, giving rise to a similar blue flame, so both the nitric oxide and the ozone, which are combining along the tube to give rise to the air afterglow, react with the carbon bisulphide. The spectrum of the blue glow is continuous, but shows none of the diffuse bands previously mentioned. Sulphur is also deposited upon the glass walls.

The extinction of the air afterglow by water and organic vapours is attributed to oxidation of these vapours without

the production of any glow or flame. As the pressure of the admitted vapour increases, the nitric oxide and ozone are used up in the oxidation process and so disappear as constituents for the formation of the air afterglow.

The oxygen afterglow at a pressure of 0.2 mm. of mercury is decidedly more greenish in colour than that of the air glow, but, as shown in Pl. VI., gives a continuous spectrum extending from λ 6800– λ 4200, although over the range λ 5000– λ 5500 the intensity is less than that over the rest of the spectrum. Superimposed upon the continuity are faint diffuse bands similar to those in the air afterglow. Microphotometer records indicate that the wave-lengths of these bands are identical with those in the air afterglow, so that the emitter giving rise to them must be the same in the two cases. Vapours of benzene, acetone, and water quenched the oxygen afterglow, whereas carbon sulphide vapour when mixed with the afterglow gave the characteristic blue flame, similar to that obtained with the air glow. All the results obtained, therefore, point to the fact that both oxygen and air afterglows arise from the oxidation of nitric oxide by ozone, both of these gases being formed in the discharge-tube. The product thus formed gives rise to the faint diffuse bands. Unfortunately the latter are so masked by the continuous spectrum that a detailed study is not practicable.

The microphotometer records were photographed by Prof. A. E. Owen, of Bangor, to whom the author expresses his sincere thanks.

LX. *Fractional Calculus.*

By W. FABIAN, *M.A., Ph.D.* *

1. *Introduction.*

LEIBNITZ† was the first to give a meaning to a fractional differential coefficient of a function. The subject has since then been investigated by many writers, and various definitions, mostly leading to the same

* Communicated by the Author.

† ‘Opera, ed. Dutens, 3, *Commercium philos. et Math.*’ p. 105 (1695).

form, have been given for the fractional integral. Ferrar* gives a full discussion of the various definitions used.

The definition adopted in the present paper is a generalized form of Riemann's definition †, which is the one now usually taken. Riemann arrived at the definition

$$D^{-\lambda} f(x) = \frac{D^\gamma}{\Gamma(\lambda + \gamma)} \int_0^x (x-t)^{\lambda+\gamma-1} f(t) dt,$$

where λ is real and γ an integer such that $\lambda + \gamma > 0$, by assuming an expansion of $f(x+h)$ in the form

$$f(x+h) = \sum_{n=-\infty}^{\infty} k_n \left(\frac{d}{dx} \right)^{n+\nu} f(x) \cdot h^{n+\nu}, \quad [n+\nu = -\lambda],$$

with the condition that this series should become a Taylor series for integral values of λ . Hardy and Littlewood ‡ and Hadamard § have used the same form for the fractional integral for real values of λ and rectilinear paths of integration in the complex plane. Although $D^{-\lambda} f(x)$ has been used sometimes || for complex values of λ , no writer has as yet attempted to give a theory of the general form $\left(\frac{d}{dz} \right)^{-\lambda} f(z)$, where λ may be

real or complex, and the integration may be performed along any simple curve in the complex plane.

Taking for this generalized λ th integral of $f(z)$ the form

$$D^{-\lambda} f(z) = \frac{D^\gamma}{\Gamma(\lambda + \gamma)} \int_a^z (z-t)^{\lambda+\gamma-1} f(t) dt,$$

where γ is the least integer greater than or equal to zero such that $\text{R}(\lambda) + \gamma > 0$, we give four index-laws. The first has been used by Levy ¶ for the real variable and real values of λ . The second is new. The third and fourth index-laws are extensions of the first. We then proceed to find conditions under which a complex function $f(z)$ can be expanded in the series which

* Ferrar, Proc. Edin. Math. Soc. xlviii. pp. 92-105 (1927).

† Riemann, 'Gesammelte Werke,' pp. 331-344 (1876).

‡ Hardy and Littlewood, Math. Zeitschrift, xxxiv. pp. 403-439 (1931).

§ Hadamard, Journ. de Math. (4) viii. pp. 101-186 (1892).

|| For instance, by Grünwald, Zeitschrift, für Math. und Phys. xii. p. 441 (1867).

¶ Levy, Bull. des Sciences Mathématiques, xlvii. p. 307 (1923).

Riemann assumed for a function of a real variable as a basis for deriving a meaning of $\left(\frac{d}{dx}\right)^{-\lambda} f(x)$, and we give an example of such an expansion.

2. Fractional Integration and Differentiation.

Let l be a simple curve. We define a λ th integral, or a $(-\lambda)$ th differential coefficient, of $f(z)$ along l as

$$D^{-\lambda}(l_a) f(z) = \frac{D}{\Gamma(\lambda + \gamma)} \int_a^z (z-t)^{\lambda+\gamma-1} f(t) dt,$$

where the integration and differentiation are along l , starting from a ; λ is any number, real or complex; γ is the least integer greater than or equal to zero such that $R(\lambda) + \gamma > 0$, $R(\lambda)$ being the real part of λ ; and D stands for $\frac{d}{dz}$, denoting differentiation along l . a is arbitrary and independent of z . $f(z)$ is bounded on l , and is continuous on l except possibly at a finite number of points on any finite arc of l . The integration is in the sense of Riemann.

When no ambiguity can arise we shall write $f_\lambda(z)$ for $D^{-\lambda}(l_a) f(z)$. Unless otherwise stated, λ and a are to be taken as finite.

If two fractional integrals $f_\lambda(z)$ and $f_\mu(z)$ occur in an operation, those values of $f_\lambda(z)$ and $f_\mu(z)$ are to be taken, for which $\text{Log}(z-t)$ in

$$(z-t)^\lambda = e^{\lambda \text{Log}(z-t)}$$

and

$$(z-t)^\mu = e^{\mu \text{Log}(z-t)}$$

is in each case the sum of its principal value and the same multiple of $2\pi i$.

The following preliminary theorem is necessary:—

THEOREM 1.—*For all values of λ such that $R(\lambda) > 0$, $f_\lambda(z)$ is continuous on l .*

Proof.—In the t -plane, let σ be a line of continuity of $f(t)$ on l . Let p and q be two points on σ between its end-points. If z be any point on σ between or at p and q , let h be any quantity such that the points $z+h$

and $z-h$ lie on σ . We have

$$\begin{aligned}
 f_{\lambda}(z+h)-f_{\lambda}(z) &= \frac{1}{\Gamma(\lambda)} \int_a^p \{(z+h-t)^{\lambda-1}-(z-t)^{\lambda-1}\} f(t) dt \\
 &\quad + \frac{1}{\Gamma(\lambda)} \int_p^{z+h} (z+h-t)^{\lambda-1} f(t) dt - \frac{1}{\Gamma(\lambda)} \int_p^z (z-t)^{\lambda-1} f(t) dt \\
 &= \frac{1}{\Gamma(\lambda)} \int_a^p \{(z+h-t)^{\lambda-1}-(z-t)^{\lambda-1}\} f(t) dt \\
 &\quad + \frac{1}{\Gamma(\lambda)} \int_{p-h}^z (z-t)^{\lambda-1} f(t+h) dt - \frac{1}{\Gamma(\lambda)} \int_p^z (z-t)^{\lambda-1} f(t) dt \\
 &= \frac{1}{\Gamma(\lambda)} \int_a^p \{(z+h-t)^{\lambda-1}-(z-t)^{\lambda-1}\} f(t) dt \\
 &\quad + \frac{1}{\Gamma(\lambda)} \int_p^z (z-t)^{\lambda-1} \{f(t+h)-f(t)\} dt \\
 &\quad + \frac{1}{\Gamma(\lambda)} \int_{p-h}^p (z-t)^{\lambda-1} f(t+h) dt,
 \end{aligned}$$

from which it can be seen that $f_{\lambda}(z)$ is continuous at the points of continuity of $f(t)$ for variations of z along l .

If c be a point of discontinuity of $f(t)$ on l , let the two parts, into which c divides l , be l_1 and l_2 , where l_1 is that arc which contains a . Let $F(t)$ be a function such that

$$F(t)=f(t) \text{ on } l_1$$

and

$$F(t)=f(t)+M \text{ on } l_2,$$

where M is a constant such that $F(t)$ is continuous at c for variations of t along l , or has at most a removable discontinuity at this point :

$$f_{\lambda}(z)=F_{\lambda}(z)$$

for points z on l_1 , and

$$f_{\lambda}(z)=F_{\lambda}(z)-\frac{1}{\Gamma(\lambda)} \int_c^z (z-t)^{\lambda-1} M dt$$

for z on l_2 .

$f_{\lambda}(z)$ can now be seen to be continuous at the discontinuities of $f(t)$ also, and the proof of the theorem is completed.

THEOREM 2.—If $R(\lambda) > 0$, $R(\mu) > 0$, then

$$D^{-\lambda}(l) D^{-\mu}(l_a) f(z) = D^{-\mu}(l) D^{-\lambda}(l_a) f(z) = D^{-\lambda-\mu}(l_a) f(z).$$

Proof.—

$$\begin{aligned} D^{-\lambda}(l_a)D^{-\mu}(l_a)f(z) &= \frac{1}{\Gamma(\lambda)} \int_a^z (z-t)^{\lambda-1} f_{\mu}(t) dt \\ &= \frac{1}{(\Gamma(\lambda)\Gamma(\mu))} \int_a^z (z-t)^{\lambda-1} dt \int_a^t (t-\tau)^{\mu-1} f(\tau) d\tau \\ &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_a^z f(\tau) d\tau \int_{\tau}^z (t-\tau)^{\mu-1} (z-t)^{\lambda-1} dt \\ &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \cdot \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} \int_a^z f(\tau) \cdot (z-\tau)^{\lambda+\mu-1} d\tau \\ &= D^{-\lambda-\mu}(l_a)f(z). \end{aligned}$$

Similarly,

$$D^{-\mu}(l_a) D^{-\lambda}(l_a) f(z) = D^{-\lambda-\mu}(l_a) f(z).$$

THEOREM 3.—If $f_{\lambda}(z)$ and $f_{\lambda+\gamma+m}(z)$ exist, where m is any integer such that $m+\gamma \geq 0$, then

$$f_{\lambda}(z) = D^{\gamma+m} f_{\lambda+\gamma+m}(z).$$

Proof.—

I. Let $m > 0$. We have

$$\begin{aligned} f_{\lambda}(z) &= D^{\gamma+m} D^{-m} f_{\lambda+\gamma}(z) \\ &= D^{\gamma+m} f_{\lambda+\gamma+m}(z), \text{ by Theorem 2.} \end{aligned}$$

II. For the case $m < 0$,

$$\begin{aligned} f_{\lambda}(z) &= D^{\gamma+m} D^{-m} f_{\lambda+\gamma}(z) \\ &= D^{\gamma+m} D^{-m} D^{\delta} f_{\lambda+\gamma+\delta}(z) \end{aligned}$$

by (I.) above, where δ is the least integer greater than or equal to zero such that

$$R(\lambda) + \gamma + m + \delta > 0;$$

hence

$$\begin{aligned} f_{\lambda}(z) &= D^{\gamma+m} D^{\delta-m} f_{\lambda+\gamma+\delta}(z) \\ &= D^{\gamma+m} f_{\lambda+\gamma+m}(z) \end{aligned}$$

by case I.

THEOREM 4.—Let $f(z)$ be analytic on l . Let λ and μ be two numbers, real or complex. If $R(\mu) \leq -1$, let

$$f(a) = f'(a) = \dots = f^{(\delta-2)}(a) = 0,$$

where δ is the least integer greater than or equal to zero, such that $R(\mu) + \delta > 0$. Then

$$D^{-\lambda}(l_a)D^{-\mu}(l_a)f(z) = D^{-\lambda-\mu}(l_a)f(z).$$

Proof.—

I. Let $R(\mu) > 0$.

$$\begin{aligned} D^{-\lambda}(l_a)D^{-\mu}(l_a)f(z) &= D^{\gamma}D^{-\lambda-\gamma}(l_a)D^{-\mu}(l_a)f(z) \\ &= D^{\gamma}f_{\lambda+\mu+\gamma}(z), \text{ by Theorem 2,} \\ &= f_{\lambda+\mu}(z), \text{ by Theorem 3.} \end{aligned}$$

II. Now let $R(\mu) < 0$.

Integrating $f_{\mu}(z)$ by parts δ times, we get

$$\begin{aligned} D^{-\lambda}f_{\mu}(z) &= D^{\gamma}D^{-\lambda-\gamma} \left\{ \frac{f^{(\delta-1)}(a)}{\Gamma(\mu+\delta)} (z-a)^{\mu+\delta-1} + f_{\mu+\delta}^{(\delta)}(z) \right\} \\ &= \frac{f^{(\delta-1)}(a)}{\Gamma(\lambda+\mu+\delta)} (z-a)^{\lambda+\mu+\delta-1} + D^{\gamma}f_{\lambda+\mu+\gamma+\delta}^{(\delta)}(z), \\ &\quad \text{by Theorem 2,} \\ &= \frac{f^{(\delta-1)}(a)}{\Gamma(\lambda+\mu+\delta)} (z-a)^{\lambda+\mu+\delta-1} + f_{\lambda+\mu+\delta}^{(\delta)}(z). \quad (1) \end{aligned}$$

Integrating $f_{\lambda+\mu}(z)$ by parts δ times, we obtain

$$\begin{aligned} f_{\lambda+\mu}(z) &= \frac{f^{(\delta-1)}(a)}{\Gamma(\lambda+\mu+\delta)} (z-a)^{\lambda+\mu+\delta-1} + f_{\lambda+\mu+\delta}^{(\delta)}(z) \\ &= D^{-\lambda}f_{\mu}(z), \text{ by (1).} \end{aligned}$$

The following theorem follows immediately from the theorem just proved :—

THEOREM 5.—Let $f(z)$ be analytic on l . Let λ and μ be two numbers, real or complex, and ρ the least integer greater than or equal to zero such that

$$R(\lambda) + \rho > 0, \quad R(\mu) + \rho > 0.$$

If $\rho \geq 2$, let $f(a) = f'(a) = \dots = f^{(\rho-2)}(a) = 0$. Then

$$D^{-\lambda}(l_a)D^{-\mu}(l_a)f(z) = D^{-\mu}(l_a)D^{-\lambda}(l_a)f(z) = D^{-\lambda-\mu}(l_a)f(z).$$

3. The Riemann Series*.

THEOREM 6.—Let $f(z)$ be analytic on l . Let a , c , and z_0 be three different points on l , taken in this order on this curve. If

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\{ D^{-\lambda-n}(l_c)f_{-\lambda-n}(z_0) \right. \\ \left. + \frac{1}{\Gamma(\lambda-n)} \int_a^c (z_0-t)^{\lambda-n-1} f_{-\lambda+n}(t) dt \right\} = 0, \end{aligned}$$

* See the Introduction to this paper.

where the integrations are along l , and a is the lower limit of integration for $f_{-\lambda-n}(z_0)$ and $f_{-\lambda+n}(t)$, then

$$f(z_0) = \sum_{n=-\infty}^{\infty} \frac{D^{\lambda+n}(l_a)f(c)}{\Gamma(\lambda+n+1)} (z_0-c)^{\lambda+n}.$$

Proof.—All integrations in this proof are along l . Unless otherwise indicated, the lower limit of integration is a .

Let β be such that $0 \leq R(\beta) < 1$, and such that $\lambda + \beta$ is an integer.

By Theorem 4

$$\begin{aligned} f(z_0) &= D^{\beta} f_{\beta}(z_0) \\ &= \frac{D}{\Gamma(1-\beta)} \left(\int_a^c + \int_c^{z_0} \right) (z_0-t)^{-\beta} f_{\beta}(t) dt \\ &= \sum_{n=-m}^{m-1} \frac{f_{\beta-n}(c)}{\Gamma(-\beta+n+1)} (z_0-c)^{-\beta+n} \\ &\quad + \frac{1}{\Gamma(-\beta-m)} \int_a^c (z_0-t)^{-\beta-m-1} f_{\beta+m}(t) dt \\ &\quad + \frac{1}{\Gamma(-\beta+m)} \int_c^{z_0} (z_0-t)^{-\beta+m-1} f_{\beta-m}(t) dt \end{aligned}$$

on integrating

$$\int_a^c (z_0-t)^{-\beta} f_{\beta}(t) dt$$

and

$$\int_c^{z_0} (z_0-t)^{-\beta} f_{\beta}(t) dt$$

by parts m times.

Hence, if

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\{ \frac{1}{\Gamma(-\beta-m)} \int_a^c (z_0-t)^{-\beta-m-1} f_{\beta+m}(t) dt \right. \\ \left. + \frac{1}{\Gamma(-\beta+m)} \int_c^{z_0} (z_0-t)^{-\beta+m-1} f_{\beta-m}(t) dt \right\} = 0, \end{aligned}$$

then
$$f(z_0) = \sum_{n=-\infty}^{\infty} \frac{f_{\beta-n}(c)}{\Gamma(-\beta+n+1)} (z_0-c)^{-\beta+n}.$$

Putting $\beta + \lambda = n - m$, we obtain the expansion enunciated in the theorem, with the conditions in the form given there.

Example.—To expand $\frac{(z-a)^p}{p!}$, where p is a positive integer, in a Riemann series.

Using the notation of the last theorem, we have

$$\begin{aligned} f(z) &= \frac{(z-a)^p}{p!}, \\ f_{-\lambda-n}(t) &= \frac{(t-a)^{-\lambda-n+p}}{\Gamma(-\lambda-n+p+1)}, \\ f_{-\lambda+n}(t) &= \frac{(t-a)^{-\lambda+n+p}}{\Gamma(-\lambda+n+p+1)}, \\ f_{-\lambda-n}(c) &= \frac{(c-a)^{-\lambda-n+p}}{\Gamma(-\lambda-n+p+1)}. \end{aligned}$$

Hence, when

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\{ \frac{1}{\Gamma(\lambda-n)} \int_a^c \frac{(z-t)^{\lambda-n-1} (t-a)^{-\lambda+n+p}}{\Gamma(-\lambda+n+p+1)} dt \right. \\ \left. + \frac{1}{\Gamma(\lambda+n)} \int_c^z \frac{(z-t)^{\lambda+n-1} (t-a)^{-\lambda-n+p}}{\Gamma(-\lambda-n+p+1)} dt \right\} = 0, \end{aligned}$$

$$\text{then } \frac{(z-a)^p}{p!} = \sum_{n=-\infty}^{\infty} \frac{(c-a)^{-\lambda-n+p} (z-c)^{\lambda+n}}{\Gamma(\lambda+n+1) \Gamma(-\lambda-n+p+1)}.$$

To find points at which this series converges to $\frac{(z-a)^p}{p!}$,

choose c such that $|a-c|=1$. The above expansion will be shown to hold at all points z on the circle of centre c and unit radius such that the lines joining z and c make with the line joining c and a angles greater than or equal to $\frac{\pi}{2}$.

If z be such a point in the t -plane satisfying these conditions, and the path l of integration is taken along the lines joining a and c , and c and z , we have, for $n > n_0 \geq \text{Re}(\lambda) + \gamma$,

$$\left| \int_a^c (z-t)^{\lambda-n-1} (t-a)^{-\lambda+n+p} dt \right| < \int_a^c |dt|.$$

From the relation

$$\Gamma(\lambda-n) \Gamma(-\lambda+n+1) = \frac{\pi}{\sin \pi(\lambda-n)}$$

we have

$$\lim_{n \rightarrow \infty} \frac{1}{\Gamma(\lambda - n)\Gamma(-\lambda + n + p + 1)} = 0.$$

Hence

$$\lim_{n \rightarrow \infty} \int_a^c \frac{(z-t)^{\lambda-n-1}(t-a)^{-\lambda+n+p}}{\Gamma(\lambda-n)\Gamma(-\lambda+n+p+1)} dt = 0.$$

Similarly

$$\lim_{n \rightarrow \infty} \int_c^z \frac{(z-t)^{\lambda+n-1}(t-a)^{-\lambda-n+p}}{\Gamma(\lambda+n)\Gamma(-\lambda-n+p+1)} dt = 0.$$

The expansion obtained holds, therefore, at all points of the semicircle considered here.

LXI. *The Critical Conditions of Dynamical Systems and Elastic Structures.* By W. J. DUNCAN, D.Sc., A.M.I. Mech.E., Wakefield Lecturer on Aeronautics at University College, Hull*.

1. Introduction and Summary†.

THE object of this paper is to draw attention to a method for determining the critical conditions of structures which was first given by C. Minelli‡. Minelli states that the correctness of his method, which is explained below, may be regarded as axiomatic, and he supplies no proof. However, it is now shown that the method only leads to correct results when the forces acting are conservative, a condition which is usually not satisfied in aerodynamical problems. Nevertheless, the method is convenient in application and is valid for a large class of cases.

Consider a dynamical system or elastic structure which is simultaneously subject to the action of two sets of forces, one constant and the other proportional to a variable parameter α (for example, in aerodynamical problems α may be the square of the air speed). Then there may be a critical value or values of α for which the static stability of the system is neutral. When in

* Communicated by E. F. Relf, A.R.C.Sc.

† The author is indebted to the Aeronautical Research Committee for permission to publish this paper.

‡ *L'Aerotecnica*, January 1934, p. 3.

α . In a finite displacement the work done against the forces of elasticity is

$$E = \frac{1}{2} E_{11} q_1^2 + \frac{1}{2} E_{22} q_2^2 + E_{12} q_1 q_2 + \text{etc.}, \dots \quad (2)$$

while the work done by the forces which are proportional to α is

$$W = \alpha \left\{ \frac{1}{2} F_{11} q_1^2 + \frac{1}{2} F_{22} q_2^2 + \frac{1}{2} (F_{12} + F_{21}) q_1 q_2 + \text{etc.} \right\}, \quad (3)$$

$$= \alpha F, \text{ say.} \quad (4)$$

In equations (1) put $P_1 = P_2 = \dots = P_n = 0$, multiply the equations by $q_1, q_2, \dots, q_n$ respectively (which need not be zero for a critical value of α), and add. The result can be written

$$E = \alpha F \quad (5)$$

Now let this equation be used to *define* α , and let arbitrary small increments δq be given to the coordinates q . Then

$$F \delta \alpha = \delta E - \alpha \delta F = \sum_{r=1}^n \left(\frac{\partial E}{\partial q_r} - \alpha \frac{\partial F}{\partial q_r} \right) \delta q_r \quad (6)$$

Put all the generalized forces P equal to zero, multiply equations (1) by $\delta q_1, \delta q_2, \dots, \delta q_n$ respectively, and subtract them from (6). Then

$$F \delta \alpha = \frac{1}{2} \alpha \sum_{i \neq j} (F_{ij} - F_{ji}) q_j \delta q_i = \frac{1}{2} \alpha \sum_{i < j} (F_{ij} - F_{ji}) (q_j \delta q_i - q_i \delta q_j), \quad (7)$$

where the last summation extends to all values of j , and to all values of i less than j . The elasticities E_{ij} do not appear in (7) since $E_{ij} = E_{ji}$. Now if the forces which are proportional to α possess a potential, then

$$F_{ij} = F_{ji}, \quad (8)$$

and equation (7) shows that α is stationary. But if these forces cannot be derived from a potential, then in general α is not stationary.

3. Application to obtain the Torsional Divergence Speed of a Cantilever Wing.

Consider a fore-and-aft section of the wing. If a normal downward load be applied near the leading edge the wing will twist nose down, whereas if the load be applied near the trailing edge the twist will be reversed. There is one intermediate point of application, known as the flexural centre, which is such that there is no twist

of the wing at the section where the load is applied. In general, a load applied at the flexural centre of the selected section will cause twist at other sections. However, in the remainder of this discussion it will be assumed that the wing structure is such that a normal load applied at the flexural centre in any section causes no twist at other sections. When this is so, the twist of the wing is entirely determined by the moments of the applied forces about the flexural centres of the sections in which they are applied. Let these moments be defined as torsional moments. Then the twist of the wing when placed in an air-current will be determined by the gravitational and aerodynamical torsional moments. Now the complete set of aerodynamical forces on the wing do not form a conservative system, but the torsional aerodynamical forces taken by themselves are conservative*. Hence Minelli's method can be used to obtain the critical value of the air speed, that is, the torsional divergence speed for the wing.

The following is a formal proof of the correctness of the method in this case.

Let y = distance of a typical section from the wing root,

s = span from root to tip,

θ = total angle of twist at the section,

n = elastic torsional stiffness per unit length,

V = air speed,

$MV^2\theta$ = aerodynamical twisting moment per unit length,
where n and M will in general be functions of y .

Then the elastic energy per unit length is $\frac{1}{2}n\left(\frac{d\theta}{dy}\right)^2$,

and the work done by the aerodynamical moments per unit length is $\frac{1}{2}MV^2\theta^2$. Hence at the critical divergence speed

$$V^2 \int_0^s M \theta^2 dy = \int_0^s n \left(\frac{d\theta}{dy} \right)^2 dy. \quad \dots \quad (9)$$

Suppose that θ receives the variation $\delta\theta$. Then (9) gives

$$V^2 \int_0^s M \theta \delta\theta dy + V \cdot \delta V \int_0^s M \theta^2 dy = \int_0^s n \frac{d\theta}{dy} \frac{d\delta\theta}{dy} dy,$$

* This is only strictly correct if the wing sections are aerodynamically independent.

and if V is stationary for small arbitrary variations of θ this becomes

$$V^2 \int_0^s M \theta \delta \theta dy = \int_0^s n \frac{d\theta}{dy} \frac{d\delta\theta}{dy} dy. \quad (10)$$

$$\text{But } \int_0^s n \frac{d\theta}{dy} \frac{d\delta\theta}{dy} dy = \left[n \frac{d\theta}{dy} \delta\theta \right]_0^s - \int_0^s \frac{d}{dy} \left(n \frac{d\theta}{dy} \right) \delta\theta dy,$$

and the expression in square brackets vanishes since $\delta\theta=0$ when $y=0$, and $n \frac{d\theta}{dy}=0$ when $y=s$, for this expresses the fact that the torsional couple at the wing tip is zero. Hence (10) becomes

$$\int_0^s \left\{ MV^2\theta + \frac{d}{dy} \left(n \frac{d\theta}{dy} \right) \right\} \delta\theta dy = 0,$$

and since this must be satisfied for all small variations $\delta\theta$ it follows that

$$MV^2\theta + \frac{d}{dy} \left(n \frac{d\theta}{dy} \right) = 0. \quad (11)$$

But this is precisely the differential equation for θ , and the correctness of the method is proved.

LXII. Transient Whirling of a Rotor.

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1. Introduction.

IF the c.g. of every section of a rotor, taken at right angles to the axis of rotation, lies on that axis, the rotor is running true. It is whirling when the c.g. of the

* Communicated by the Author.

whole rotor, or of any portion of it bounded by planes normal to the axis of rotation, has a motion round that axis.

The whirl is a steady one when the c.g. continually repeats the same motion in the same time ; it is a transient whirl when it gradually decays because of friction, or grows under the action of some driving force.

A simple whirl is one whose radius revolves uniformly ; the c.g. moves steadily round a circle with a steady simple whirl, and in some sort of spiral with a transient one. Other whirls are compound ones and may be regarded as the resultant of two or more simple whirls, each with its own rotation.

The rotation of a whirl is the angular velocity of its radius * ; the rotation of the unbalance whirls is the same as that of the shaft, but in other cases the whirl rotation may have no connexion with the shaft speed, and may even revolve in the opposite way.

In order to simplify matters the following assumptions must be made :—

- (a) The elastic force opposing any lateral deflexion of the shaft is in line with, and proportional to, the deflexion of the c.g. from some stated position.
- (b) The frictional force opposing the whirl is proportional to the velocity of the c.g. and directly opposite to it.
- (c) The mass of the rotor is concentrated at one point in it.
- (d) The speed of the shaft is kept constant unless otherwise stated.

Except for one or two very simple problems the first two assumptions are absolutely necessary to make any useful mathematical analysis possible, and the third is required to make that analysis reasonably simple.

If hysteretic effects can be neglected, the first assumption may be taken as accurate so long as the strain in the material is kept below the elastic limit. The second is a good approximation with small whirl speeds, but the square law would be better with larger ones if it were possible to deal with it mathematically.

Distributed inertia greatly complicates the mathe-

* Note that in this paper "rotation" means angular velocity, not angular displacement.

matical work. It permits several modes of whirling, each with its own critical speed, whereas concentrated mass gives only one. Otherwise the third assumption does not seriously affect the general character of the results.

When the first two assumptions are valid the Principle of Suppression can be applied. This principle states that, when calculating the additional motion due to an additional set of external forces, we may suppress entirely the original external forces if at the same time we suppress the motion whose elastic, frictional, and inertia forces together balance these original external forces, and take as the zero of displacement the moving point of live equilibrium at which that balance is obtained *.

The constants involved in the assumptions (a) and (b) are defined by the following relations :—

$$\begin{aligned} \text{Elastic force per unit mass of rotor} \\ = \omega_0^2 \times \text{deflexion of c.g.} \quad . \quad . \quad . \quad . \quad (1.01) \end{aligned}$$

$$\begin{aligned} \text{Frictional force per unit mass of rotor} \\ = 2\alpha \times \text{velocity of c.g.} \quad . \quad . \quad . \quad . \quad (1.02) \end{aligned}$$

Both ω_0 and α are angular velocities, the first being the critical speed of the whirl and the second the damping velocity.

The reciprocal of α is a time, known as the damping time-constant. It is the time taken by the transient to decay to $1/e$, or 36.79 per cent. of its original value. After intervals of 2, 3, 4, and 5 times $1/\alpha$ the radius has shrunk to 13.53, 4.98, 1.83, and 0.67 per cent. respectively of its initial value. After four or five times the time-constant the vibration may be considered as having practically ceased.

When a rotor is capable of visible transverse vibrations a rough idea of the magnitude of α may be obtained by observing the duration of the vibrations which follow a transverse blow, and dividing this time into 3 or 4. The shaft may be running or at rest, but in the former case stroboscopic observation may be necessary to enable the transient to be picked out from the other motion.

The following symbols are employed in the paper :—

a = Eccentricity due to unbalance.

r = Radius of unbalance or other steady whirl.

* A full discussion of the Principle of Suppression was given by the author in 'The Engineer' (see Bibliography, no. (9)).

r_0 = Radius of unbalance whirl at the critical speed.

ρ = Radius of transient whirl.

ρ_i = Initial value of same.

ρ_f, ρ_b = Its forward and backward components.

v_t = Velocity of transient motion at the initial instant.

$\Delta\rho_i$ = Same, but relative to a plane rotating with the shaft.

v = Total velocity of c.g.

R = Radius of guard ring.

s = Radius of shaft.

t = Time.

x, y = Coordinates of c.g. referred to axes rotating with the shaft.

j = "Turn forward 90° " or " 90° ahead of."

ω_0 = Critical speed = (elastic force per unit mass of ideal rotor divided by deflexion of c.g.) $^{\frac{1}{2}}$.

α = Damping constant = frictional ditto divided by twice velocity of c.g.

ω = Rotation of shaft.

ω_t = Rotation of transient whirl.

ω_f, ω_b = Rotation of forward and backward components relative to shaft *.

ω_v = Phase velocity (pulsatance) of free transverse vibrations of shaft.

ω_g = Rotation corresponding to frequency of flashes or glimpses.

ϕ = Angle of lag of radius of unbalance whirl behind neutral radius.

ϕ_t = Same for radius to actual position of c.g.

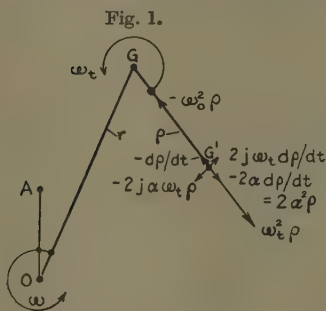
2. *Transient Whirls.*

A transient occurs whenever the conditions are suddenly changed, for then the c.g. is no longer at the place where it should be for steady equilibrium under the new conditions and/or it no longer has the same motion as the new point of equilibrium. Thus, with the unbalance whirl, transients

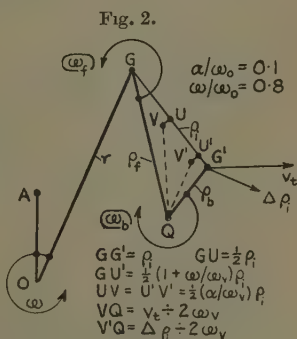
* On the diagrams symbols denoting relative rotations are ringed round.

arise when the speed of the shaft is quickly altered or when some constraint upon the position of the shaft is suddenly removed, and also, of course, after any jar or blow.

The transient corresponds to the addition of a new external force of zero magnitude. It follows from the Principle of Suppression, when that is valid, that the transient is exactly the same whether the shaft is running or at rest, and whether the original equilibrium was a steady deflexion due to, say, gravity, or was the true-running condition, or a simple steady whirl due to any cause and having any rotation, or a compound whirl having any number of components.



Transient whirl.



Components of the transient whirl to meet initial conditions.

At every instant, including the initial instant with which we are more immediately concerned, the transient must make up for the difference between the actual displacement and that corresponding to the steady whirl, and also for the discrepancy between the velocities.

In fig. 1 let G be the position of equilibrium for the steady whirl, say that due to an unbalance OA . G' is the actual position of the c.g. at the same instant. G' travels round G and at the same time approaches it.

Suppressing the motion and forces associated with the steady whirl, we are left with the rotation round G , and have only to take account of deflexions from that point. The radius of the transient whirl is $GG' = \rho$, and its rotation is ω_t .

Omitting the suppressed forces which balance at G the following forces remain, their magnitudes being given per unit mass :—

Radial.

Elastic force

$$-\omega_0^2 \rho. \quad . \quad . \quad . \quad . \quad (2.01)$$

Centrifugal force

$$+\omega_t^2 \rho. \quad . \quad . \quad . \quad . \quad (2.02)$$

Inertia force opposing change of radial shrinkage velocity

$$-d^2 \rho / dt^2. \quad . \quad . \quad . \quad . \quad (2.03)$$

Frictional force opposing change of radius

$$-2\alpha d\rho / dt. \quad . \quad . \quad . \quad . \quad (2.04)$$

Tangential.

Friction opposing tangential motion

$$-2j\alpha\omega_t \rho. \quad . \quad . \quad . \quad . \quad (2.05)$$

Inertia force opposing Coriolis acceleration

$$-2j\omega_t d\rho / dt. \quad . \quad . \quad . \quad . \quad (2.06)$$

The operator j means “ turn forward 90° ” or “ 90° ahead of ”; hence

$$j^2 = -1, \quad \text{and} \quad 1/j = -j.$$

Since the tangential forces must balance, we have

$$2j\omega_t d\rho / dt + 2j\alpha\omega_t \rho = 0, \quad . \quad . \quad . \quad . \quad (2.07)$$

or

$$d\rho / dt = -\alpha \rho. \quad . \quad . \quad . \quad . \quad (2.08)$$

Whence

$$d^2 \rho / dt^2 = -\alpha d\rho / dt = \alpha^2 \rho, \quad . \quad . \quad (2.09)$$

and

$$\rho = \rho_0 e^{-\alpha t}. \quad . \quad . \quad . \quad . \quad (2.10)$$

Equating the positive and negative radial forces we get

$$\omega_t^2 \rho = \omega_0^2 \rho + d^2 \rho / dt^2 + 2\alpha d\rho / dt \quad . \quad . \quad (2.11)$$

$$= (\omega_0^2 - \alpha^2) \rho. \quad . \quad . \quad . \quad (2.12)$$

Hence

$$\omega_t = \pm (\omega_0^2 - \alpha^2)^{\frac{1}{2}} = \pm \omega_v. \quad . \quad (2.13)$$

We thus see that the absolute rotation of the transient whirl may be in either direction and that it has the same magnitude as ω_v , the pulsatace or phase velocity of the free vibration of the shaft as a beam, which equality,

of course, we also get directly from the Principle of Suppression.

For most purposes we may take this phase velocity as identical with the critical speed ω_0 , but where the difference between them is of any consequence its amount is

$$(\omega_0 - \omega_v) = (\omega_0^2 - \omega_v^2) / (\omega_0 + \omega_v) \quad . \quad . \quad (2.14)$$

$$\doteq \alpha^2 / 2\omega_0 = \frac{1}{2}(\alpha/\omega_0)^2 \omega_0. \quad . \quad . \quad (2.15)$$

With $\alpha/\omega_0 = 0.1$ the difference is 0.5 per cent. Usually the friction is less than this and the difference between these speeds is under one part in a thousand.

3. Initial Magnitude of Transient.

At every instant the displacement of the c.g. is the resultant of that due to the steady whirl and that due to the transient, and the same applies to the velocities. Consequently at the initial instant the transient whirl must have a radius equal to the distance of the c.g. from the point of live equilibrium for the steady whirl, and it must also make up for the difference between the actual velocity of the c.g. and that corresponding to the steady whirl.

Since the radius is fixed by the position of G' a simple transient whirl would have only two possible velocities, corresponding to the two rotations $+\omega_v$ and $-\omega_v$; the second condition could only be satisfied if the discrepancy velocity happened to agree with one of these special values.

But if GG' be the vector sum of two radii, one with the rotation $+\omega_v$ and the other with $-\omega_v$, any required velocity can be obtained by a suitable choice of the two component radii, ρ_f and ρ_b .

Thus the transient will practically always be a double one with components having equal and opposite rotations; each will decay in accordance with the factor $e^{-\alpha t}$ as required by equation (2.10).

Suppose that at the initial instant the c.g. is at G' in fig. 2, and that it has a velocity relative to G of v_i in the direction shown. Denoting GG' by ρ_i , we have the following vector relationships:—

$$\rho_f + \rho_b = {}_v\rho_i^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.01)$$

* The symbol $=_v$ denotes an equation which is true vectorially and not true algebraically. When necessary to avoid confusion $+_v$ and $-_v$ are also used for vectorial addition or subtraction, and $=_a$, $+_a$, $-_a$ for algebraic equality, addition, and subtraction.

$$v_t = {}_v j \omega_v (\rho_f - \rho_b) - \alpha (\rho_f + \rho_b). \quad . \quad . \quad . \quad (3.02)$$

$$\therefore \rho_f - \rho_b = {}_v - j (\alpha / \omega_v) \rho_i - j v_t / \omega_v. \quad . \quad . \quad . \quad (3.03)$$

$$\therefore \rho_f = {}_v \frac{1}{2} \rho_i - \frac{1}{2} j (\alpha / \omega_v) \rho_i - \frac{1}{2} j v_t / \omega_v. \quad . \quad . \quad (3.04)$$

It is sometimes convenient to deal with the velocity of the transient relative to a plane rotating about G at the same speed as the shaft. This relative velocity is

$$\Delta \rho_i = {}_v v_t - j \omega \rho_i, \quad . \quad . \quad . \quad (3.05)$$

where ω is the rotation of the shaft.

Equation (3.04) then becomes

$$\rho_f = {}_v \frac{1}{2} (1 + \omega / \omega_v) \rho_i - \frac{1}{2} j (\alpha / \omega_v) \rho_i - \frac{1}{2} j \Delta \rho_i / \omega_v. \quad . \quad (3.06)$$

In fig. 2, U is the midpoint of GG'. Set off UV backwards at right angles to GG' equal to $\frac{1}{2} (\alpha / \omega_v) \rho_i$, and then draw VQ 90° behind v_t and equal to $\frac{1}{2} v_t / \omega_v$. GQ is the required forward component of the transient and QG' the backward one. This construction merely carries out the vector process represented by the right-hand side of equation (3.04).

When the velocity is given relative to the plane rotating about G at shaft speed we must use equation (3.06). GU' is now $\frac{1}{2} (1 + \omega / \omega_v) \rho_i$, U'V' is the same as UV, and V'Q is at right angles to the relative velocity and equal to that velocity divided by $2\omega_v$.

The total velocity of the c.g. is the resultant of the transient motion and that due to the rotation of G around O. When OG rotates along with the shaft, as it does with the unbalance whirl, we have

$$v = {}_v v_t + j \omega r = {}_v \Delta \rho_i + j \omega (r + \rho_i). \quad . \quad . \quad (3.07)$$

Now when the c.g. is at O, the vector sum of the steady and transient radii is zero; hence in this particular case only $\Delta \rho_i$ is the total velocity of the c.g., and it is zero when the c.g. is at rest at the centre.

4. Form of Transient Motion.

Since the two components of the transient whirl have equal and opposite rotations in space, their resultant, in the absence of damping, is a motion round an ellipse whose centre is at G and moves with G. The axes of the ellipse have fixed directions lying along the bisectors of the angles between the component radii and have lengths equal to the sum and difference of these radii.

The way of the rotation round the ellipse is that of the longer radius.

With equal radii the elliptical motion degenerates into a harmonic vibration along a straight line whose direction is fixed in space. When one component radius is zero the ellipse becomes a circle whose radius is the other component.

If a uniform motion round a circle be projected on to an oblique plane we get a non-uniform motion round an ellipse whose major axis is equal to the diameter of the circle and whose minor axis is smaller in the ratio of the cosine of the angle of obliquity.

Now let the radius of the circle shrink at a rate proportional to its own length at the same time as it revolves uniformly. The circle becomes an equiangular spiral and the ellipse a peculiar form of spiral whose radius waxes and wanes during each revolution but diminishes from revolution to revolution.

It is a motion of this kind which is given by the resultant of the two shrinking whirls with equal but opposite rotations. The angular velocity of the radius of the projecting equiangular spiral is ω_v , and all the radii shrink in accordance with the factor $e^{-\alpha t}$.

The absolute motion of the c.g. is the resultant of this shrinking elliptical motion and of the motion of G, but it is of less interest, and much less simple, than the relative transient motion of the c.g. with respect to a plane rotating with the shaft.

The relative rotations of the transient radii are $(\omega_v - \omega)$ forwards for ρ_f and $(\omega_v + \omega)$ backwards for ρ_b . The forward radius always has a forward rotation of ω_v in space, but relative to the shaft it rotates backwards when the shaft speed exceeds ω_v .

In the absence of friction the relative motion of the c.g. is thus some form of trochoid, including the cycloids as special cases; the particular kind of trochoid depends upon the initial conditions. Any trochoid may be regarded as an ellipse whose axes revolve at the mean speed of the component radii whilst it is being traced out, and it is actually seen as a revolving ellipse under suitable conditions*. With friction we have a shrinking trochoid of some kind.

* A full discussion of these curves will be found in the article previously mentioned (see no. (9) in Bibliography).

The following relationships are of use when plotting the form of the relative motion due to the transient whirl :—

Relative rotation of forward radius

$$\omega_f = (\omega_v - \omega). \quad . \quad . \quad . \quad . \quad . \quad (4.01)$$

Relative rotation of backward radius

$$\omega_b = -(\omega_v + \omega). \quad . \quad . \quad . \quad . \quad . \quad (4.02)$$

Rotation of forward radius relative to backward one

$$2\omega_v. \quad . \quad . \quad . \quad . \quad . \quad (4.03)$$

Time taken to trace one lobe of trochoid

=time interval between alternate meetings of radii

=time for one revolution of ρ_b relative to ρ_f ,

$$\pi/\omega_v. \quad . \quad . \quad . \quad . \quad . \quad (4.04)$$

Angular displacement of ρ_f relative to shaft in this time

$$\pi(\omega_v - \omega)/\omega_v. \quad . \quad . \quad . \quad . \quad . \quad (4.05)$$

Ditto for backward radius ρ_b

$$-\pi(\omega_v + \omega)/\omega_v. \quad . \quad . \quad . \quad . \quad . \quad (4.06)$$

Angular displacement of shaft in this time

$$\pi(\omega/\omega_v). \quad . \quad . \quad . \quad . \quad . \quad (4.07)$$

Angular displacement of ρ_f relative to shaft during one revolution of shaft

$$2\pi(\omega_v - \omega)/\omega. \quad . \quad . \quad . \quad . \quad . \quad (4.08)$$

Ditto for ρ_b

$$-2\pi(\omega_v + \omega)/\omega. \quad . \quad . \quad . \quad . \quad . \quad (4.09)$$

Number of times ρ_b passes ρ_f for one relative revolution of ρ_f

$$\{2\pi/(\omega_v - \omega)\} + (\pi/\omega_v) = 2\omega_v/(\omega_v - \omega). \quad . \quad . \quad (4.10)$$

5. *Examples of Transient Whirls.*

There is an unlimited variety of possible transients, corresponding to the infinite number of ways in which the c.g. might be located and be moving at the initial instant; but a few examples will suffice to show their general character.

In model experiments the condition most easily realized is the sudden release of the shaft when held in the true running position. Here the c.g. starts from rest at O,

the axis of rotation, and this case has been chosen for figs. 4-10, in each of which OG (OB in the no-friction cases) is the steady unbalance whirl radius due to an eccentricity OA; OA is referred to as the "neutral radius."

Fig. 3 gives the general case and has been drawn for the same constants as fig. 2, which is, in fact, a portion of fig. 3 drawn separately.

Beginning with GQ, set off a number of positions of the forward radius, GQ₁, GQ₂, etc., at convenient intervals in the forward direction when ω_f is positive, or in the backward direction when the shaft speed exceeds the critical speed. Compared with its original length each radius has to be reduced in the ratio

$$\epsilon^{-\alpha t} = \epsilon^{-(\alpha/\omega_f)\omega_f t}, \quad . \quad . \quad . \quad . \quad . \quad (5.01)$$

where $\omega_f t$ is the angle between the forward radius and its original position.

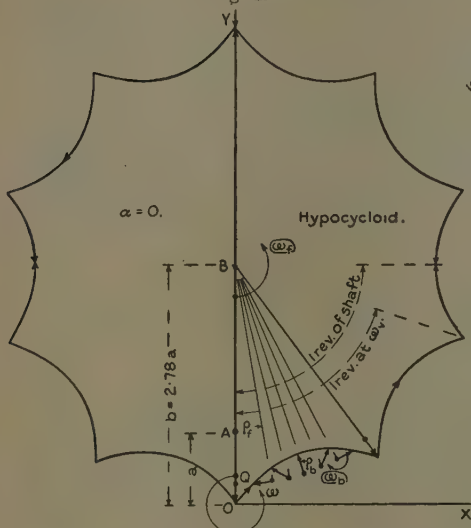
The corresponding backward radii, after reduction in the same ratio as the forward ones, are now set off from the centres Q₁, Q₂, etc.; their direction has to be set back from the original one by ω_b/ω_f times the angle turned through by the forward radius, the rotation being backward whether ω_f be positive or negative.

One lobe of the trochoid corresponds to one revolution of the backward radius relative to the forward one, and hence to $\omega_b t = \pi$. It is convenient to take the spacing between successive positions of the forward radius a submultiple of the angle subtended by the lobe at the centre.

Instead of continuing to work from the two component radii, after the first lobe has been drawn it is rather quicker to get the other lobes by copying the first, the total radius from G being measured and reduced in the proper ratio to get the radius to the corresponding point in the next lobe.

The simplest way to take account of the shrinkage is to plot along a horizontal line to some convenient scale the values of $\epsilon^{-\alpha t}$ for the selected angles, and to erect ordinates at each point of division. The initial length of the radius is then set up along the ordinate for $t=0$ and its extremity joined to the other end of the horizontal line, where $t=\infty$. The intercepts on the ordinates between the horizontal and sloping lines give the lengths

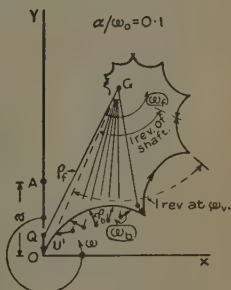
Fig. 4.



No friction.

Fig. 5.

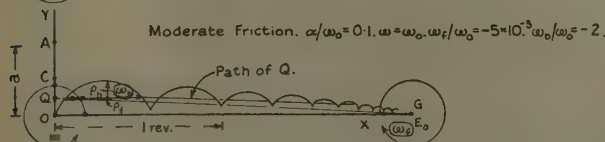
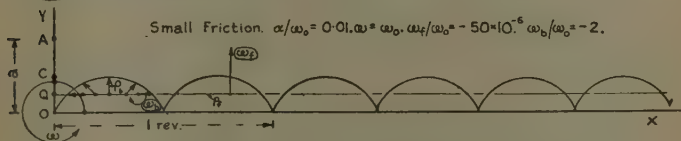
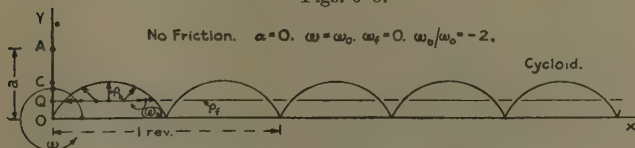
$$\begin{aligned}\omega/\omega_0 &= 0.8. \\ \omega_f/\omega_0 &= 0.2. \\ \omega_b/\omega_0 &= -1.8.\end{aligned}$$



Moderate friction.

Relative path of c.g. when released from rest at centre with shaft running below the critical speed.

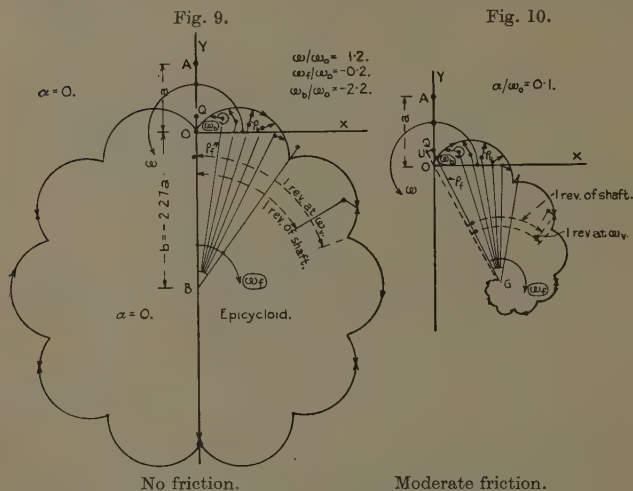
Figs. 6-8.



Relative path of c.g. when released from rest at centre with shaft running at the critical speed.

Observe that the rotation round G , relative to the shaft, is forward; this will always be the case for speeds below ω_v when the c.g. starts from rest at the centre, and serves as a delicate test as to whether the speed is above or below the critical speed when it is close to it.

For figs. 6-8 the shaft is running at the critical speed. The point of equilibrium for the unbalance whirl is at E_0 and lies 90° behind the neutral radius, the distance OE_0 being $(\omega_0/2\alpha)a$. When the motion starts with the c.g. at O , E_0O is the value of ρ_i . We shall consider this case



Relative path of c.g. when released from rest at centre with shaft running above the critical speed.

by itself in the next section, as it has many points of interest.

For the last pair of curves, figs. 9-10, the shaft runs at 20 per cent. above the critical speed. The forward radius is now rotating backwards relative to the shaft, and the relative rotation of the c.g. round the point of equilibrium is backwards.

Fig. 9 gives the no-friction case, and the curve is an epicycloid. The number of lobes for one convolution is only an integer for certain speeds, of which one has been selected for the diagram.

Fig. 10 is the shrinking epicycloid for this speed when the friction is the same as in figs. 2, 3, 5, and 8.

6. Transient with Shaft running at the Critical Speed.

With the shaft running at the critical speed, $\omega = \omega_0$, and the equilibrium radius for the unbalance whirl is OE_0 , 90° behind OA (fig. 8). The deflexion of the shaft due to this whirl is

$$AE_0 = AO + OE_0 = a + r_0. \quad (6.01)$$

The elastic force due to the deflexion component OE_0 is now equal and opposite to the centrifugal force, whilst that due to the other component AO balances the friction. Hence

$$2\alpha\omega_0 r_0 = \omega_0^2 a \quad \text{or} \quad r_0 = (\omega_0/2\alpha)a. \quad (6.02)$$

If the shaft be steadied at the centre and then released the transient radius is initially $\rho_i = E_0O$ and $\Delta\rho_i = 0$. Remembering that ω_v is identical with ω_0 when there is no friction, and very little different from it with a moderate amount of friction, the construction for equation (3.06) gives the following values:—

$$E_0U' = \frac{1}{2}(1 + \omega_0/\omega_v)r_0 = r_0 = (\omega_0/2\alpha)a. \quad (6.03)$$

Thus U' lies exactly at O when there is no friction and very close to O when there is friction. Also

$$U'V' = \frac{1}{2}(\alpha/\omega_0) \times (\omega_0/2\alpha)a = \frac{1}{4}a. \quad (6.04)$$

Since $U'V'$ lies 90° behind E_0O , it lies 90° ahead of OE_0 , and therefore along OA. But Q coincides with V' , since the velocity is zero. Hence Q lies in OA at a quarter of its length from O.

Thus the component radii of the transient following the release of the shaft from the centre whilst running at the critical speed are

$$\rho_f = E_0Q = E_0O = (\omega_0/2\alpha)a. \quad (6.05)$$

$$\rho_b = \frac{1}{4}a. \quad (6.06)$$

With no friction $(\omega_v - \omega) = 0$ at the critical speed, and the direction of ρ_f is fixed relatively to the shaft. The relative path of the c.g. following its release from the centre is a cycloid.

But when friction is present $(\omega_v - \omega_0)$ is not quite zero and ρ_f has the small backward rotation $\alpha^2/2\omega_0$

(see eq. (2.15)). The backward rotation of ρ_b may be taken as $2\omega_0$ without serious error.

The rotation of ρ_f raises its Q-end, but the shrinkage lowers it. To begin with, these actions just balance, and Q starts off parallel to OE_0 ; but after a little the shrinkage predominates and Q approaches OE_0 , as may be seen in fig. 8.

At first the shrinkage of ρ_b causes the cusps to rise slowly, but later the drop of Q is more than enough to compensate for this; the cusps then approach OE_0 again, and finally converge on E_0 .

The cycloid of fig. 6 extends to infinity. In fig. 7 OE_0 is fifty times OA and only about one-sixth of it appears on the diagram. In it, the rise of the cusps is just visible, but the drop of Q is not. With the larger friction of fig. 8 the whole of OE_0 comes into view, and both these features can be seen.

The depth of the lobe is never more than $\frac{1}{2}a$, which is very small compared with OE_0 with the usual amounts of friction. Hence the path of the c.g. when released from the centre with the shaft running at the critical speed does not depart greatly from the straight line OE_0 , which, of course, is rotating along with the shaft.

When ρ_b is negligible the velocity along OE_0 is equal to the rate of shrinkage of ρ_f , namely

$$\alpha\rho_f = \frac{1}{2}\omega_0 a. \quad . \quad . \quad . \quad . \quad (6.07)$$

Taking rotating coord'nates along OE_0 and OA, the exact equations for the motion of the c.g. can easily be written down from the geometry of fig. 8. We shall write down only the simpler ones for the no-friction case of fig. 6; these are

$$x = \frac{1}{2}a\omega_0 t - \frac{1}{4}a \sin 2\omega_0 t, \quad . \quad . \quad . \quad (6.08)$$

$$y = \frac{1}{4}a(1 - \cos 2\omega_0 t). \quad . \quad . \quad . \quad (6.09)$$

A case of some theoretical interest, but which could hardly be realized experimentally, is that in which, with the shaft running at the critical speed, the c.g. is started off from O with the velocity $\frac{1}{2}\omega_0 a$ along OE_0 . Q then falls on O and the small radius vanishes. The Coriolis acceleration with this velocity is $\omega_0^2 a$, which, is exactly that produced by the elastic force due to the

AO component of the deflexion so long as the c.g. lies in OE_0 .

At the critical speed the time of one revolution is $2\pi/\omega_0$. Hence in the special case we are here considering the radial distance of the c.g. increases by the amount πa during each revolution of the shaft.

When there is no friction the c.g. continues to move steadily along OE_0 with constant velocity, because at the critical speed the radial component of the elastic force is exactly balanced by the centrifugal force.

But when friction is present the velocity decays and the balance is disturbed; the tangential elastic force exceeds the Coriolis force and pulls the c.g. forwards, corresponding to the backward rotation of ρ_f round E_0 which has already been mentioned. The relative path then becomes a steep spiral instead of the straight line OE_0 .

To begin with, and until $(\alpha t)^2$ becomes appreciable compared with unity, the equations for this case reduce to

$$x \doteq \frac{1}{2} a \omega_0 t (1 - \frac{1}{2} \alpha t) \quad . \quad . \quad . \quad . \quad (6.10)$$

$$y \doteq \frac{1}{4} a \alpha t (1 - \alpha t) \quad . \quad . \quad . \quad . \quad (6.11)$$

Since the transient corresponds to the addition of zero external force, it will always appear as part of the complete solution of *any* whirling problem which assumes the conditions required by the Principle of Suppression. Several writers have obtained the equivalent of equations (6.10)–(6.11), but referred to fixed axes, as the solution of special problems without recognizing that they refer only to the transient and to a very special case of that.

When the shaft speed is ω_c instead of ω_0 the forward radius does not rotate relatively to the shaft, but only shrinks in accordance with the damping factor. Hence the centre of the small radius moves more and more slowly along QG. But G is no longer at E_0 ; it is at a distance of $\frac{1}{4}a$ from E_0 and directly ahead of it.

Taking the other constants as in fig. 8, G would be level with the position where Q is shown. OQ would be perpendicular to the new direction of OG, and thus a little ahead of OA. Except that it ends at the new position of G the path would be little different from fig. 8. The cusps of the curve would lie exactly on the straight line joining the new position of G to O and the tops of the lobes on that joining it to C.

7. *Experimental Observation of the Transient.*

When the transient is superposed on the unbalance whirl, or on the circular motion of the observation mark due to its own eccentricity, the total whirl radius waxes and wanes as the two whirls get into and out of phase. The frequency of these oscillations is equal to the difference between the frequency of rotations of the two whirls—that is, between ω_v and ω .

If the speed of the shaft be not more than about 200 r.p.m. away from its critical value, and if the rotor be sufficiently flexible to permit them to be easily seen, the number of these oscillations of amplitude in a measured time can be counted and their frequency determined. Then ω_v is obtained by adding this frequency to, or subtracting it from, the shaft speed, according as the latter is below or above the critical speed.

When this test is carried out at a number of shaft speeds it is found that the result is always the same within the limits of experimental error.

With speeds more remote from the critical, stroboscopic observations must be employed, preferably using an electrically maintained vibrator to control the frequency of the flashes.

Stroboscopic illumination eliminates, so far as vision is concerned, all periodic motions whose frequency is a multiple or submultiple of the flash frequency. Other periodic motions are seen at a reduced frequency equal to the difference between their actual frequency and that of the flashes or of some multiple or submultiple of it.

The vibrator must first be tuned to the free vibrations of the shaft as a beam when not rotating. When the adjustment is correct the stroboscopic appearance of the shaft after plucking to one side is a slow return to the centre *without vibration*.

It will be found that the stroboscopic appearance of the transient, so long as it can be distinguished from the other motion, remains exactly the same when the shaft is rotating.

The unbalance whirl running at the shaft speed is not eliminated, but is seen at reduced speed. Near the critical speed this whirl may be too large to permit the transient to be examined closely; but within this range the first

method is at its best, and between the two methods the constancy of the transient rotation can be proved over the whole speed range from zero upwards.

A slotted card attached to a small motor can be used instead of the vibrator, but its constancy of speed is very much inferior.

The old "hatpin experiment" for demonstrating the unbalance whirl also shows up the transient very prettily. The pin, which should preferably have a white head, is mounted in a chuck on a lathe or small motor, and the speed, or the projecting length of the pin, is adjusted until a whirl of about 1 to 2 inches diameter is obtained.

Transients can be started by steadying the pin at the centre and then releasing it, or by flipping it gently. The path of the head is seen as a continuous trace and takes the form of a waxing and waning circle.

Many very curious figures, such as triangles and figures of eight, can be obtained by placing the fingers to act as stops limiting the motion in certain directions.

For studying the relative transient motion the apparatus already described by the author may be employed*, using a round shaft instead of a flatted one.

The model consists of a $\frac{1}{4}$ " steel shaft $36\frac{3}{8}$ " long, with 35.2" clear between the bearings, and carrying an ebonite disk $2\frac{1}{2}$ " diameter and $\frac{1}{2}$ " thick. The shaft weighs 230 grams, and the disk with its brass hub 67 grams. A small white circle on the disk, $1\frac{1}{16}$ " from its centre, serves for an observation mark.

The shaft runs in brass bearings, originally $\frac{3}{8}$ " long, but afterwards reduced to 0.10". The critical speed is about 740 r.p.m. with the final arrangement, but was higher in some of the earlier ones.

The shaft is driven by a 5 h.p. D.C. motor through a pin coupling consisting of a pin fixed to the motor shaft, projecting into a slot in a flange fixed to the model. A spring prevents chattering by holding the pin against the same side of the slot as the driving force does.

The motor is, of course, very large for the power used by the model, but in earlier experiments it was found impossible to hold the speed sufficiently constant with a fractional h.p. motor. Moreover, the auxiliaries available

* Fig. 11, p. 213 of no. (8) in Bibliography.

for the large machine enabled its speed to be set at any value from zero to twice the critical speed of the model, and to be held steady with great accuracy.

A mirror at 45° to the shaft was fixed behind it and a clear glass parallel to the shaft in front of it. A ruled card in front of the glass was adjusted until its image in the clear glass fell at the same place as the image of the observation mark in the mirror. In this way a scaled background was obtained against which the motion of the mark could be seen and roughly measured.

The ruling on the card consists of ten concentric circles 0.1" apart and a number of straight lines radiating out from their common centre.

The maximum safe deflexion of the shaft is over one inch, and a strong guard ensured that it shall not be bent more than this.

A card with a single slot is held in a chuck on the motor; light from an electric lantern flashes through this slot once per revolution, so as to give stroboscopic illumination of the model with shaft frequency.

With stroboscopic illumination at shaft frequency all motion at that frequency is obliterated, leaving the transient alone visible, unless there are other whirls whose rotation does not agree with that of the shaft.

The direction of rotation, relatively to the shaft, of the transient which follows release from the centre gives a delicate test as to whether the speed is below (transient seen going forwards) or above (transient backwards) the critical speed.

We do not, however, see the details of the transient, nor even that the relative path of the c.g. is a lobed curve, because the observation mark is only seen intermittently once per revolution of the shaft.

Crosses have been marked on figs 3, 4, 5, 9, 10 to show the position of the c.g. at successive flashes, on the supposition that the first coincides with the release of the shaft from the centre.

No crosses have been put on figs. 6-8, in which they would come at alternate cusps. In an actual case the loops are shallow, but they would not be seen with stroboscopic vision at the critical frequency, because that agrees with the transient rotation.

Although these crosses are fairly far apart the separate glimpses do blend into a continuous picture over a con-

siderable range of speed on each side of the critical, especially when the transient is small.

It is only at certain exceptional speeds that the crosses come at the same point on each convolution ; they usually fall into the gaps between the previous ones and so assist the delusion of continuous motion.

When the speed is too far from the critical value only a flutter is visible with stroboscopic vision at shaft frequency, especially when the amplitude of the transient is large or the speed low.

The motion, relative to a plane revolving with the shaft, of all points on the central plane of the disk is identical with that of the c.g. Hence when studying this motion it is not necessary to place the observation mark at the centre of the shaft, which, indeed, would not be possible save with an overhung shaft, but it may be put at any convenient position on the disk.

8. Observed Behaviour of Transient.

Using the stroboscope with shaft frequency it is most fascinating to watch the transient which follows the release of the shaft from the centre, or any other disturbance, especially near the critical speed. The spot is seen to travel round the point of equilibrium in a converging spiral, forwards or backwards according to the shaft speed.

With the shaft running at its critical speed the shaft moves out very gently along a straight line 90° behind the eccentricity. Its velocity gradually diminishes until it finally comes to rest at the point of equilibrium if the balance be good enough to keep that point within the permitted limit of deflexion. The least departure from ω_c is shown by rotation of the spot round the equilibrium position.

With constant illumination the spot traces out a white circle with a steady whirl. When a small transient is added this circle is seen to wax and wane as the relative phases change ; but with a large transient the trace is seen as a sort of double spiral, a converging half for the inward motion and a diverging one for the outward movement.

Many discrepancies will be found between the observations and the simple theory given above, because of the factors which have been left out of account.

Elastic hysteresis and allied effects, due to the friction at the clamping fits, coupling pins, etc., are apt to be troublesome when the shaft runs above the critical speed, for they then give rise to a tangential force driving the whirl forwards against its own friction.

If this driving force grows with increasing amplitude more rapidly than the whirl friction the condition may become unstable, and the shaft cannot be run above the critical speed for any length of time. It was noticed that these effects were more troublesome with the short bearings than with the old ones, but that may have been due to a change of bearing clearance or to the accumulation of residual strain in the shaft.

These hysteretic effects are rather erratic, and depend upon minor changes of condition which are not under control. The behaviour of the model with one set-up may be quite different from that with an earlier one seemingly identical with it, and the same happenings may not be repeated after steadying the shaft.

Here are a few typical examples to show the behaviour of the model when running above the critical speed. A small transient may die away, whereas a larger one may grow until the guard is struck. The transient may grow to about 0.5 in. radius, and be steadily maintained at that radius. When running at a high speed the shaft may run true for a while; then a tiny transient begins to grow slowly until it has a radius of about $\frac{1}{4}$ in., after which it increases more and more rapidly until the guard is struck.

Another discrepancy arises from the increase of stiffness and critical speed when the bearings begin to grip the journals. Normally the shaft ends are free to tilt about the inboard edges of the bearings, but when the bending is sufficient to make the outboard ends of the journals cross the bearing clearances that freedom is taken away and the shaft becomes a beam with fixed ends. The transition is gradual because of the elastic yielding of the surfaces of contact.

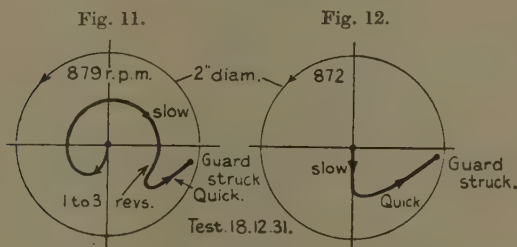
A static test on the original "long" bearings, which were worn out to a large clearance, showed a constant stiffness up to about 0.5 in. deflexion; beyond that the stiffness gradually increased until the critical speed was raised by 50 per cent., when the deflexion was one inch.

The short bearings were designed to avoid this change

of stiffness, but when they were new the stiffness was only constant up to about 0.75 in. After they were worn a bit this limit would be raised.

In both cases the linear limit was found to be lower for transient whirls, partly because the transient could indicate a smaller change of stiffness than the static test, and partly because the distribution of the bending force along the shaft would be different in the two tests.

Collars used to locate the shaft endwise produce a similar effect as soon as the tilt of the shaft ends is sufficient to make the collars press against the bearings. This was eliminated by inserting long fine helical springs between the collars and the bearings. These springs can give sufficient control of the endwise location without appreciably changing the lateral stiffness.



Transients when stiffness increases with amplitude.

Examples of such transients are given in the sketches of figs. 11 and 12. The endwise location was controlled by the springs and the bearings were the short ones newly made with very little clearance.

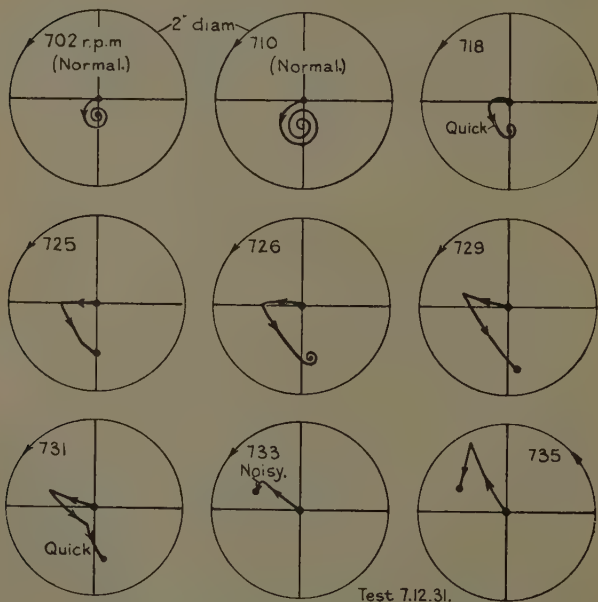
In fig. 11 the speed is just over the critical speed for small amplitudes. At first the transient revolves backwards as it grows, but when the bearings begin to nip the shaft at about 0.5 in. deflexion the critical speed rises above the shaft speed and the relative rotation of the transient reverses. In this case the transient grew until the guard was struck; the last portion of the path was traced out quickly, showing that there had been a considerable increase of the critical speed.

In fig. 12 the shaft is running exactly at the initial critical speed. The initial relative motion of the c.g. is straight outwards until the critical speed begins to rise,

after which the path bends forwards and the subsequent behaviour is similar to the other example.

This change of critical speed with amplitude causes the unbalance whirl to keep on increasing when the speed is raised above the initial critical value, because the critical speed keeps running away from you as the amplitude grows; but if the shaft be steadied it will continue to run quietly close to the centre.

Fig. 13.



Irregular transients due to fouling by setscrew.

There are thus two possible positions of equilibrium for the unbalance whirl at speeds between the lowest and highest points of the range over which the critical speed changes (about 1 : 2 in the limit), depending upon whether the large amplitude obtained on first approaching the critical speed has been damped out or not.

Irregularities also arise when some projection on the

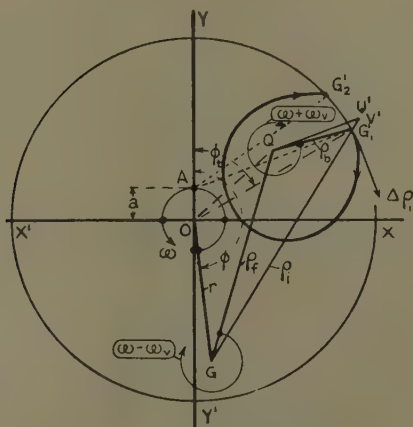
rotor runs clear normally but fouls some fixed part when the shaft is sufficiently bent.

Fig. 13 shows some examples of this kind caused by the head of the setscrew on the coupling rubbing against a part of the pedestal; all were taken on the same run. In this case one locating spring only was used, and it held the coupling against the bearing; the bearings were the "long" ones and the disk had a felt backing intended to increase the air friction opposing the whirl.

9. Repeated Transient due to Striking Guard.

When the guard is struck by the shaft, vibrations of great interest are produced under certain conditions.

Fig. 14.



Transient on rebound from guard.

In fig. 14 the large circle marks the boundary of the region within which the centre of the shaft is permitted to move. Its radius is $(R-s)$, where R is the radius of the guard ring and s is that of the shaft.

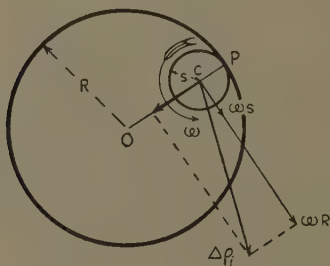
Suppose the shaft to be running above its critical speed, and that the equilibrium position of the c.g. is at G , the neutral radius being OA as usual. Suppose, further, that because of some previous transient the shaft has struck the guard, and that the c.g. is at G_1' at the instant of rebound.

Because the guard has held the shaft back during the contact, the c.g. will acquire a backward velocity relative to the plane rotating about O at shaft speed, and it will also have some radial velocity of rebound.

In the extreme case the point of the shaft surface which is touching the guard will be at rest tangentially at the instant of leaving the guard. This condition is shown on fig. 15. Since P has no tangential motion the centre of the shaft must be moving backwards with the velocity ωs , whereas it would have a forward velocity of $\omega (R-s)$ if it were fixed on the revolving plane.

On leaving the guard the maximum tangential velocity relative to this plane is thus ωR , but it will usually be less

Fig. 15.



Rebound of shaft from guard.

than this maximum because of slipping along the guard. Combining this with the radial velocity of rebound, we get $\Delta_{0\rho_i}$ as the velocity of the c.g. relative to the plane revolving about O at shaft speed.

Before applying the construction of fig. 2 we must first shift the centre of rotation to G by adding $-j\omega r$ to get the $\Delta\rho_i$ of that formula. We can thus obtain GQ as the forward transient and QG_1' as the backward one.

Since the shaft is running above its critical speed both transients are rotating backwards relative to the shaft; these backward rotations are $(\omega - \omega_v)$ and $(\omega + \omega_v)$ respectively. The relative motion of the c.g. is thus a shrinking epitrochoid, but with the usual amount of friction the shrinkage during the time which concerns us at present is almost negligible.

Provided the transient be large enough, the c.g. will again reach out to the limiting circle after a time not greatly different from $2\pi/(\omega + \omega_v)$. The shaft again strikes the guard and a similar action is repeated.

This happens at G_2' , close to G_1' on the diagram revolving with the shaft; but of course G_1' and G_2' are opposite entirely different portions of the fixed guard.

The angle between the two positions at which the guard is struck is of the order of $2\pi\omega/(\omega + \omega_r)$. Some particular shaft speed will make that angle exactly 180° , and it will not differ greatly from that value at any speed near the critical speed.

If G_2' be ahead of G_1' , in the direction of the shaft rotation, and if during contact the c.g. be delayed exactly by the distance $G_2'G_1'$, the second transient will be identical with the first, and the same action will be repeated indefinitely.

The new transient is not a continuation of the old one but a fresh one started at each collision, the period of the cycle being a little less than π/ω_r . This phenomenon is therefore not a sustained transient but a series of identical transients following one another at regular intervals.

Some kinetic energy is lost at each rebound, and there is a further loss from friction during the free motion; but during contact the shaft is bent sidewise, and there is an increase of potential energy whose amount is proportional to $(AG_1'^2 - AG_2'^2)$; it must be equal to the losses if the second transient is to be the same as the first.

Now

$$AG'^2 = OA^2 + OG'^2 - 2OA \cdot OG' \cos \phi_t, \quad (7.01)$$

$$d(AG'^2)/d\phi_t = 2OA \cdot OG' \sin \phi_t. \quad (7.02)$$

Hence the increase of potential energy by a given small delay distance $G_2'G_1'$ is a maximum when G' lies on OX , 90° behind OA , and is zero when G' lies on YOY' . On the other half of the circle the sign is reversed; the transient there loses potential energy as well as kinetic energy at each collision, and would therefore quickly cease to cause striking.

If the energy of the new transient be more than that of the original one the transient will be bigger and G_2' will lie ahead of the first point of contact. Consequently

less energy will be received at its next collision than at the former one, and the following transient will be somewhat smaller. The point G_2' will thus gradually work forwards until it reaches the position in which the energy imparted at each blow is just the right amount to make the new transient identical with the preceding one.

If the transient be too small, G_2' will work backwards, and a steady state will be attained provided that the maximum energy available when G_2' reaches OX is sufficient. If not, G_2' passes OX, and then the transient receives less and less energy as G_2' falls farther behind. After a few collisions the shaft will fail to strike the guard and the transient will ultimately die away.

The energy given is proportional to the eccentricity OA. Consequently a certain minimum eccentricity is necessary to produce a repeated transient.

Fig. 14 is drawn for a rotor running just above its critical speed, in which case GQ moves slowly backwards relatively to OA. By this motion Q approaches the limiting circle, and thereby assists the backward whirl to reach the limiting circle in spite of their joint decay. The production of a repeated transient is thus promoted.

As the speed is raised GQ revolves more quickly (relatively), which of itself would increase the outward motion of Q, but at the same time G goes nearer to O, and this reduces it. Above a certain speed the outward motion of Q because of the rotation is less than the inward motion due to the shrinkage, and thus Q moves inwards. Beyond a certain point this would lead to a failure to strike the guard.

We thus conclude that raising the speed above the critical speed at first promotes the repeated transient, but afterwards antagonizes it, and finally stops it.

Below the critical speed GQ has a forward relative rotation, and at the same time G itself lies above OX. So long as G is behind OQ, Q will move inwards. The speed need only be a little under the critical value for this motion to suffice to prevent striking. Thus the repeated transient will disappear when the speed is reduced a little below the critical value.

The motion of the c.g. has three component whirls—OG with the absolute rotation $+\omega$, GQ with $+\omega_r$, and QG' with $-\omega_c$. Near the critical speed the first two combine into one, OQ, whose radius is not quite constant

and whose rotation is slightly variable about a mean value, which is a little less than ω with the shaft running above the critical speed and a little greater below it.

Approximately therefore the resultant motion consists of two whirls of opposite rotations; the path of the c.g. is then almost a hypotrochoid, which becomes an ellipse when the shaft speed exceeds the critical speed by an amount equal to the mean relative rotation of OQ.

At speeds near to this one we may regard the motion as one round an ellipse whose axes are rotating. At each instant the axes of the ellipse lie along the bisectors of the angle OQG'. Their absolute rotation is thus the mean of those of OQ and QG'; it will be forwards at speeds above that corresponding to stationary axes and backwards below it.

For the elliptical motion to degenerate into a straight line OQ and QG' would have to be alike. If the construction of fig. 14 be carefully studied it will be seen that this condition might reasonably occur when there is not too much slipping of the shaft on the guard.

10. *Experimental Observations of Repeated Transient.*

The author has frequently observed, and been puzzled by, the repeated transient due to collision with the guard, but he was never able to produce it at will with certainty. As it occurred whilst trial runs were being made to investigate some other feature the conditions were variable from run to run and were not always known accurately.

In his model the steel guard ring had an opening 2 in. in diameter, which had a thick lining of soft tape. As the guard was merely intended to act as a safety stop, no particular care was taken to centre it accurately, and it certainly was eccentric during some of the runs.

For examination of the vibrations, three kinds of illumination were employed, often simultaneously; these are: CI=constant illumination; SI=stroboscopic illumination of the shaft frequency, controlled by a slit in a disk attached to the driving motor; and VI=stroboscopic illumination controlled by an electrically maintained vibrator of the author's pattern, whose frequency was approximately the same as the critical speed of the shaft. The vibrator phase velocity is denoted by ω_v .

Some observations were taken on the shaft itself ($\frac{1}{4}$ in. diam.) and others on an observation mark on the ebonite disk forming the "wheel" of the rotor. When the shaft runs true this mark traces out a circle $2\frac{1}{8}$ in. diameter at the shaft speed, and the observed motion is the resultant of this eccentricity and the movement of the shaft centre.

With the rotor sufficiently out of balance the elliptical vibration of the shaft was obtained just above the critical speed as a residual effect of the large amplitude whilst passing through that speed, or because the sustained transient grew sufficiently for the guard to be struck. With the former cause it would not grow again after the shaft was steadied by hand, but in the latter case it would rebuild itself.

The axis of the ellipse would usually revolve slowly, either forwards or backwards according as the speed was above or below a particular value, and frequently the minor axis would be so short that the vibration was almost a plane one.

The following is one set of observations, given in the order in which they were taken. The vibrator tuning was slightly above the critical speed, and the differences between the shaft and vibrator speeds were measured directly by timing the apparent rotation of the observation mark as seen by VI.

- $\omega_g + 12$ r.p.m. Shaft vibrates in very elongated ellipse, whose axes change direction occasionally. Shaft steadied and released; vibration grew again; axes going slowly backwards. (Probably speed had dropped slightly.)
- $\omega_g + 10$ r.p.m. Axis of ellipse oscillates a little.
- $\omega_g - 10$ r.p.m. Unbalance whirl only, no transient.
- ω_g . Ditto.
- $\omega_g + 10$ r.p.m. Elliptical vibration re-established itself. Axes rotating backwards.
- $\omega_g + 18$ r.p.m. Ellipse revolves forwards at first, but settles down to a fixed position. (Probably speed dropped somewhat during observations.)
- $\omega + 33$ r.p.m. Shaft quietens down of itself. Speed run up to over twice the critical value with no tendency to start or hold a whirl.

In another test made at a later date, when the vibrator tuning was slightly different, the shaft vibrated almost in a plane and struck the guard very lightly; the plane of vibration revolved backwards at 27 r.p.m. when the shaft speed was 7.3 r.p.m. above that of the vibrator.

The vibration might or might not restart itself after steadying, as already mentioned, but it could always be reestablished by a sufficiently violent pluck when the conditions were right.

On yet another occasion, with the vibrator tuning slightly fast, the apparent track of the shaft seen by VI was an irregular triangle with the corners at the limiting circle set by the guard. At the same time the observation mark on the disk, seen, however, by SI, traced out a curve which was approximately a circle roughly concentric with the axis of rotation. This condition continued indefinitely, and could be repeated again and again on that occasion, for the whirl grew quickly whenever the shaft was released after having been steadied. This particular whirl continued, still running at or about the critical speed, when the shaft speed was raised to $2\omega_0$, so it was evidently a case of the sustained transient growing until the guard was struck.

In an earlier run on the same day as the last observation VI showed the shaft tracing a curve much the same as the hypocycloid of fig. 4, but with a greater number of lobes. At each cusp the shaft struck the guard very gently. The rotation round the curve was apparently backwards, but this was due to the stroboscopic reversal caused by the frequency of the vibrator being slightly in excess of the critical speed.

11. *Historical Notes.*

The very first paper on the whirling of shafts dealt with the transient, although not under that name. In 1869 Rankine ⁽¹⁾ considered the equilibrium of a shaft running in perfect balance but disturbed from its central position.

He assumed that the subsequent motion of the c.g. would be along a radius which rotates with the shaft, without recognizing that this motion would require a tangential force to make the direction of the radial velocity rotate along with the radius, and to increase the tangential velocity of the c.g. as it moves further out.

The need of this force was explained by Coriolis as far back as 1831, but the same error still survives and crops up afresh every now and again. Its latest reappearance was in 1931, exactly a hundred years after Coriolis' paper!

It is the source from which a surprising number of writers have deduced a fictitious critical condition at $1/\sqrt{2}$ times the critical speed.

Rankine's conclusion was that the equilibrium would be stable below the critical speed, neutral at that speed, and unstable above it.

Because of that error of Rankine's, engineers long believed that it was impossible to run a shaft above its critical speed without destroying it; the advent of the De Laval turbine twenty years later (1889) proved him to be wrong.

Föppl⁽²⁾ explained the true nature of the transient motion in 1895, when he showed that it was one round an ellipse when no friction exists. He did not include friction in his theory, but pointed out that in any actual case it would cause the transient to decay.

The fact that this elliptical motion contains a backward whirl as one component has been largely, but not entirely, overlooked until recently.

Stodola⁽⁴⁾, in 1916, was the first to include friction in the theory of the unbalance whirl, but he does not seem to have extended it to the transient. This was done by Jeffcott in 1919⁽⁵⁾ and by Rodgers in 1922⁽⁶⁾.

Since that time, various writers have repeated the theory, following the lines laid down by those mentioned, all of whom employed fixed rectangular coordinates.

So far as the author can ascertain the only previous use of a vector method is that given by Danis⁽³⁾ in 1916; his method involves a knowledge of the solution of vector differential equations, and does not include friction.

Little attention has been paid hitherto to the motion of the transient relative to a plane rotating with the shaft, and none at all to the effect of the initial conditions on the transient.

As a consequence of this neglect many statements appear as general which apply only with special initial conditions; for instance, it is commonly supposed that at the critical speed the transient motion is a uniform one along a radius which rotates with the shaft, whereas

this is true only with the particular conditions referred to above after equation (6.09).

The author first noticed the elliptical whirl caused by striking the guard in 1932; no reference to such a whirl has been found in any of the literature examined.

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LXIII. *On the Nature of Luminous Streamers from the Condensed Spark in Vacuo.* By C. F. BAREFORD, M.Sc., Ph.D., Physics Department, The University of Sheffield*.

[Plate VII.]

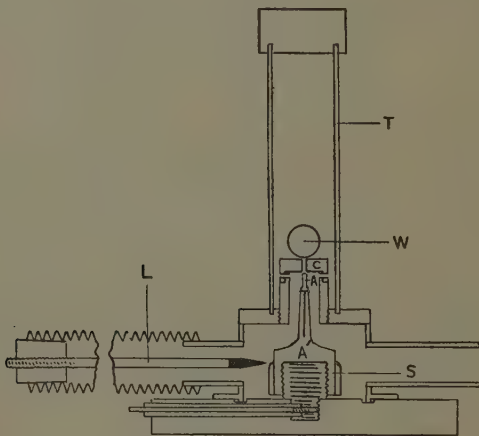
WHEN a condensed spark takes place *in vacuo* between two electrodes, one of which has a suitable hole bored through it, a luminous column or "streamer" can be observed extending from the hole into the region beyond the pierced electrode. A condensed spark *in vacuo* has been used by Millikan ⁽¹⁾ and others to extend

* Communicated by Prof. S. R. Milner, F.R.S.

spectroscopic investigation into the far ultra-violet and the discharge is known to be rich in excited, highly ionized atoms, yet little seems to be known with regard to the constitution of these streamers. The present paper records the results of some experiments made to study it.

Experimental.—The spark took place in a highly evacuated vessel T (fig. 1 a), between two electrodes A and C. A was a metallic cylinder 1 mm. in diameter and several mm. long, and C was a disk varying in thickness from 3 to 6 mm. in different experiments with a hole

Fig. 1 a.



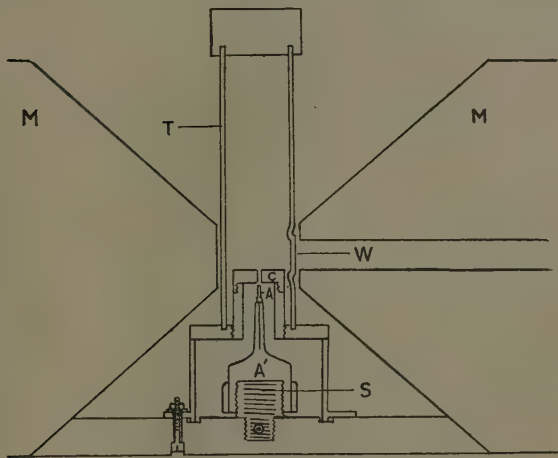
Spark chamber (side elevation).

bored through it exactly opposite A. To obtain the spark very good vacua and very high field-strengths were essential. The slightest trace of mercury vapour was found to have a deleterious action on the spark; the discharge took on a "glow" form filling all the evacuated space, and did not exhibit any sign of streamers. Consequently a Metropolitan-Vickers oil-condensation pump was used to evacuate the apparatus.

The spark was produced by the discharge of a condenser K, fig. 2, charged by an induction coil. A sufficient p.d. to produce a spark in these high vacua could seldom

be obtained when K was connected directly to the coil. Higher p.d.s at the condenser plates were obtained without increasing the power of the coil by inserting the air spark-gaps G_1 between coil and condenser. A succession

Fig. 1 b.



Spark chamber (front elevation).

Fig. 2.

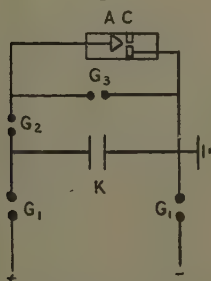


Diagram of connexion.

of sparks through these gradually raised the p.d. until the condenser discharged through the vacuum gap AC. An auxiliary air gap G_2 in series with AC helped materially in the formation of the vacuum spark by causing the

p.d. across AC to be applied suddenly. A third air gap G_2 , connected across AC, was used to estimate the potentials applied.

The field-strengths that are necessary to cause a spark, as estimated from the size of the equivalent air gap and the electrode separation, are of the order of 10^6 volts per cm. To obtain these high field-strengths it was necessary to bring A and C to within 0.2 mm. of each other. Furthermore, in view of the rapidity with which the electrodes burned away in the spark, it was also necessary to be able to adjust the distance apart of A and C at will, and to do so without destroying the vacuum. This was done by mounting A on a milled nut A', which could be moved up and down the screw S by means of a lever L. L was fixed in a flexible tombac side tube and so could easily be made to engage the millings on A'. Of the two electrodes, A was removable and could readily be changed for another metal, C was of brass or iron, and was faced with a protective disk of molybdenum to prevent its being vaporized by the spark. The molybdenum disk was pierced, and round the hole, which coincided with the canal in C, a small lip was raised projecting towards A. This was done in order to raise the field-strength between the electrodes in the region of the entrance to the canal.

With this disposition of apparatus it is found that luminous streamers, usually one or two cm. long, are projected through the canal at each spark. With clean electrodes they are produced only when A is connected to the positive plate of the condenser—connexion to the negative plate fails to produce a spark; after sparking for a short time, however, sparks and streamers may be obtained irrespective of the initial sign of the plate to which A is joined. The cause of this result seems to be as follows. The production of the spark depends primarily upon the ease with which vapour can be produced from the (initially) positive electrode A by the bombardment of it with auto-electrons, which are drawn from the cathode by the intense field in the gap. It is thus more difficult for a spark to pass when the molybdenum of C is positively charged, since this is so much less easily vaporized than, say, magnesium, which along with other soft metals was used for A. Once the molybdenum has become coated with magnesium, however, as the result

of continued sparking, there are in effect two magnesium electrodes, and the spark can then pass irrespective of the sign of A.

While a clean molybdenum electrode has a rectifying effect on the production of the initial discharge, it does not appear to prevent the spark itself from being oscillatory. That this is so was found by connecting a gas-filled discharge-tube in series with the gap AC. If the pressure in such a tube is reduced until a clearly defined cathode dark space is to be seen on the passage of a uni-directional current, then when it is placed in the spark circuit the positive column appears to fill the whole tube at each spark, and it is impossible to identify either electrode as cathode.

The Streamers.—The colour of the streamers is characteristic of the metal of A. Those from aluminium, magnesium, and silver are blue, the actual shade of blue depending on the metal; those from zinc are pale green, while those from cadmium are a deeper green. The capacity of K has a certain influence both on the colour and on the intensity. This is most marked in the case of magnesium; for a small value of the capacity the streamers are feeble and greenish, but as it is increased they become much more intense and deep blue in colour. The increase in intensity is readily explained by the increased current in the spark; the alteration of colour is due to the shorter wave-length spark lines becoming more prominent, but the cause of this is not quite so clear. Prof. Milner⁽⁴⁾ has observed that in sparks in air the potential difference falls rapidly to a limit having the arc value. Since the rate of fall will depend on the capacity of K, if the characteristic curves of sparks *in vacuo* are similar to those of sparks in air, the *average* p.d. will be higher with increased capacity, and this may be the cause of the excitation of the more ionized states which produce the spark lines.

If a glass plate is placed in the path of the streamers a metallic mirror is formed on it. This occurs even when the plate is several centimetres beyond the point where the streamer ceases to be visible. The deposit is tenacious, for although a large part of it can be rubbed off under finger pressure, the remaining traces can only be removed by treating the surface of the glass with strong acid. While fairly prolonged sparking was required to obtain

sufficiently intense deposits to be visible, it was found possible to develop the invisible traces given by only one or two sparks by the method described by Gerlach and Stern ⁽²⁾. When objects are placed in the path of the streamers sharp shadows are cast on the collecting plate, though the sharpness is considerably reduced by the use of wide canals in C.

Spectroscopic observation of the streamers was made by throwing images of them on the slit of a Hilger quartz spectrograph (E 2). The photographs show the characteristic line spectrum of the metal of A, with both spark and arc lines tailing off in intensity with increasing distance from the hole. A typical spectrogram, showing magnesium, is reproduced in fig. 3* (Pl. VII.). It was found in all cases examined that the arclines die away much more rapidly than the spark lines, a result that is in agreement with observations of L. R. Maxwell ⁽³⁾. This result for sparks *in vacuo* is interesting, because it is the exact opposite of that found by Prof. Milner ⁽⁴⁾ for the metallic streamers in oscillatory sparks in air. The cause of the difference is not clear, although it must be remembered that much stronger initial fields are requisite for sparks *in vacuo* than in air, and this may have something to do with it.

The streamers were examined in a strong transverse electric field, of the order of 1000 volts per cm., which was applied between two parallel plates placed beyond C (not shown in fig. 1a). The technique of this method of analysis is not completely developed at present, however, and it has not yielded really conclusive results. In spite of numerous precautions there is a pronounced tendency for flash-overs to take place between the deflecting plates at each appearance of the streamers. The latter do seem to be broadened out, however, in both directions along the lines of the field, that is, they behave as though there exist in them both positively and negatively charged particles. Since the spark is oscillatory, it is reasonable to infer that the luminous particles are charged (to some

* There is a possible application of this type of spectrogram to the determination of the mean life of the excited states of metallic atoms in much the same way as has already been done with luminous positive rays by Wien ⁽⁵⁾ and Kerschbaum ⁽⁶⁾. The velocities of the particles have not, however, so far been determined with sufficient accuracy to enable definite numerical results to be obtained.

extent, at any rate) alternately positive and negative in the successive oscillations.

The streamers were also examined in a transverse magnetic field, and for this purpose the whole apparatus was mounted between the poles, M, M in fig. 1 *b*, of an electromagnet. They were observed through a window W of transparent silica and through a hole bored in one pole-piece of the electromagnet. To avoid reflexions from the inner surface of the tube opposite W it was coated with a dead black paint. The spark itself was surrounded by a hollow soft iron cylinder in order to screen it from the field of the magnet. The upper end of this cylinder served as the electrode C, and was provided with a central canal as already described. When a transverse field of 15,000 gauss was applied it was found that the streamers split into two parts, proceeding in different directions as shown in fig. 4 (Pl. VII.). One (X) was deflected from the original line in the direction in which negatively charged particles would be moved, and the other (Y) in the opposite direction. No trace of an undeviated streamer was ever observed. The photographs are composite records, because the necessarily small hole in the pole-piece of the magnet restricted the aperture of the camera lens and made a hundred or more sparks essential to obtain a satisfactory plate. A close examination of X and Y shows that, in spite of the comparatively large deflexions they possess very little curvature. The composite nature of the records does, of course, tend to mask curvature, for as no two sparks follow exactly the same path no more exemplary conduct can be expected from the streamers. Examination with a microphotometer enabled a small curvature usually to be detected, however, and it was always in such a direction as to indicate a negative sign for the particles of X and a positive one for those of Y.

The magnetically deviated streamers were examined in the different monochromatic radiations of the spectrum lines which they emit. This was done by removing the slit from a quartz spectrograph and arranging the instrument so that the streamers occupied the place in which the slit formerly lay. Each image obtained in this way shows both X and Y quite independently of whether it corresponds to an arc or to a spark line. For example, fig. 5 (Pl. VII.) is a typical picture obtained when sparking

aluminium. The images *a, a* correspond to the triplets 4529–4480 Å. and 3655–3601 Å. of the spectral type Al III, while the images *b, b*, represent the lines 3962 Å. and 3092 Å. of the type Al I.

The relative intensities of the branches X and Y depend on the amount of inductance in the spark circuit. In the absence of inductance, apart from that of the straight leads, most of the metals examined were found to give a much brighter Y than X streamer. As the inductance was increased, however, X became comparatively brighter; for examples see fig. 4 (Pl. VII.), showing the streamers from magnesium, zinc, aluminium, and cadmium (*a*) with zero inductance and (*b*) with two or three microhenries in circuit. When the inductance was still further increased, however, X diminished in intensity and both X and Y became steadily more diffuse. Finally, a stage was reached when only a very diffuse Y remained as in fig. 4 (Pl. VII.) Zn(*c*), which shows the streamers of zinc in this condition. (Silver, however, fig. 4 (Pl. VII.) Ag(*a*), did not show X passing through a maximum brightness relative to Y, although otherwise it behaved as described above.)

The inclination with which X and Y leave the canal is probably due indirectly to the insufficient screening of the spark from the magnetic field. At each reversal of the oscillating discharge the magnetic field, which is too weak to influence greatly the ions during the discharge itself, swings the remaining column of nearly stationary ions (probably the electrons) to one side or the other of the gap. This determines the course of the spark in the subsequent oscillation. The sides of A (fig. 1), not its apex, alternately behave as the effective electrode, and the line of the discharge is always inclined to the vertical. The sides of A are actually found to burn down much more rapidly than the centre, and quite often a long, thin, central filament is left on it.

Nature of Carriers.—It must be admitted that at the present stage these experiments have not succeeded in determining satisfactorily the nature of the particles which form the luminous streamers. At first it seemed natural to assume that these were positive monatomic ions of the anode metal projected by the field of the spark through the hole in the cathode, and were similar

in character to the luminous canal rays that are produced in a Geissler tube when the cathode is pierced. But the later observations have raised considerable difficulties in the way of holding this straightforward view. There seems no doubt that the streamers are projected in both phases of the oscillating spark, *i. e.*, when the metal A is negative, as well as when it is positive. Further, the spectrograms in a magnetic field, such as fig. 5 (Pl. VII.), show conclusively that the spark lines as well as the arc lines are emitted by the streamers in each phase of the oscillations. According to modern theory, while the arc lines are due to neutral atoms, the spark lines require for their emission the production of ionized, positively charged atoms. How can the latter be associated with the streamers formed in the negative phase of the oscillations of the spark?

The observations in the transverse electric field were uncertain and rather erratic, but as far as they go they suggest that the streamer particles are charged alternately positive and negative in the successive oscillations. It has not been possible, unfortunately, to determine the ratio of the charge to the mass of the particles; this would require a knowledge of their velocities as well as measurements of the electric deflexion. An attempt was made to find the former by observing the streamers in a rapidly revolving mirror; but no deflexion could be observed, and the only positive result was that their velocities were greater than 5×10^5 cm./sec., the limiting value detectable. We can only infer from the observations that the particles have either high velocities or large masses. It is possible that charged particles consisting of a number of atoms are torn off from the surface of the electrode. Indeed, with a silver electrode, glowing white streaks can often be seen in the midst of a streamer, which indicate the presence of quite large incandescent particles torn from the surface. At the same time the projection of uncharged atoms by vaporization of the intensely heated surface to share in the formation of the streamer is not excluded by the observations.

The view that the streamers are particles or agglomerations of atoms would enable the emission of spark lines from the negative phase streamers to be explained, since individual atoms in a particle might well be ionized

without the particle itself being positively charged. On the other hand, it gives rise to a difficulty (perhaps a minor one) in that we should expect an atom in an agglomeration to give rise to a much more broadened line than is actually observed in the spark. It is hoped that further experiments will resolve some of these difficulties.

I should like to thank the Department of Scientific and Industrial Research, which by its grant made this work possible, and also the Trustees of the Styring Fellowship for later assistance. I should also like to express my gratitude to Professor Milner for his encouragement, advice, and many kindnesses.

ABSTRACT.

It is found that in the condensed spark in a high vacuum sharply defined luminous streamers can be obtained passing through a hole bored in one of the electrodes and extending into the exhausted space beyond it. The spectra of the streamers show the lines due to neutral and to ionized atoms of the metal of the electrode opposite to the hole. The streamers occur in both the positive and the negative phases of the oscillating current of the spark. The nature of the carriers has not been conclusively determined, but there is some reason to infer that aggregates rather than single atoms are concerned in the formation of the streamers.

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LXIV. *The Velocity of Light and the Generalized Lorentz Transformations.* By F. TAVANI*.

THE equation

$$\sum_{\nu=1}^3 \left(\frac{dx_{\nu}}{dt} \right)^2 = c^2 \quad (1)$$

is given by the Theory of Relativity for the hypothesis that the origin of the frame moves in a space of one dimension. Let us see what becomes of this equation when the space in which the origin moves, instead of one, has two or three dimensions. For this purpose we adopt the generalized transformation of Lorentz †, instead of the original transformation, in which the space of the moving origin is supposed to be of one dimension only.

Starting from the equations of the transformation in the case of the origin moving in a two-dimensional space † with a process similar to that followed for establishing them, we obtain the coordinates X, Y, Z of a point P on a three-dimensional moving frame TNB :

$$(a) \left\{ \begin{array}{ll} X = \beta_1(X' + u_1 t_1'), & X' = \beta_1(X - u_1 t_1'), \\ Y = \beta_2(Y' + u_2 t_2'), & Y' = \beta_2(Y - u_2 t_2'), \\ Z = \beta_3(Z' + u_3 t_3'), & Z' = \beta_3(Z - u_3 t_3'); \end{array} \right\} (a')$$

$$(b) \left\{ \begin{array}{ll} t_1 = \beta_1 \left(t_1' + \frac{u_1 X'}{c^2} \right), & t_1' = \beta_1 \left(t_1 - \frac{u_1 X}{c^2} \right), \\ t_2 = \beta_2 \left(t_2' - \frac{u_2 Y'}{c^2} \right), & t_2' = \beta_2 \left(t_2 - \frac{u_2 Y}{c^2} \right), \\ t_3 = \beta_3 \left(t_3' - \frac{u_3 Z'}{c^2} \right), & t_3' = \beta_3 \left(t_3 - \frac{u_3 Z}{c^2} \right). \end{array} \right\} (b')$$

Thence, by dividing each member of (a') by each member of (b'), and putting successively $X = ct_1$, $Y = ct_2$, $Z = ct_3$ in the second members of (a'), we obtain

$$\frac{X'}{t_1'} = \frac{X}{t_1} = c, \quad \frac{Y'}{t_2'} = \frac{Y}{t_2} = c, \quad \frac{Z'}{t_3'} = \frac{Z}{t_3} = c, \quad . \quad (\delta)$$

* Communicated by the Author.

† Phil. Mag., July 1934, p. 190.

a similar operation being carried between the equations (a) and (b). From the equations (δ) follow

$$\left. \begin{aligned} dX &= c dt_1, & dY &= c dt_2, & dZ &= c dt_3, \\ dX' &= c dt_1', & dY' &= c dt_2', & dZ' &= c dt_3'; \end{aligned} \right\} \quad (\epsilon)$$

thence

$$\left. \begin{aligned} &\frac{dX}{dt_1 + dt_2 + dt_3} + \frac{dY}{dt_1 + dt_2 + dt_3} + \frac{dZ}{dt_1 + dt_2 + dt_3} = c \\ \text{and} &\frac{dX'}{dt_1' + dt_2' + dt_3'} + \frac{dY'}{dt_1' + dt_2' + dt_3'} + \frac{dZ'}{dt_1' + dt_2' + dt_3'} = c. \end{aligned} \right\} \quad (\eta)$$

Both members of each equation of the group (ε) can be squared, and hence we can derive the relations

$$\sum_{\nu=1}^3 \frac{(dX_{\nu})^2}{dt_1^2 + dt_2^2 + dt_3^2} = \sum_{\nu=1}^3 \frac{(dX'_{\nu})^2}{dt_1'^2 + dt_2'^2 + dt_3'^2} = c^2,$$

comparable with

$$\sum_{\nu=1}^3 \left(\frac{dX_{\nu}}{dt} \right)^2 = \sum_{\nu=1}^3 \left(\frac{dX'_{\nu}}{dt'} \right)^2 = c^2$$

given by the Theory of Relativity in the case when the origin of the frame moves in one dimension only.

These equations show that the character of invariance of the expression of c is kept also in the case when the origin moves in a space of three dimensions. The process followed for establishing the above relations, as it is given here, as well as is given in the Theory of Relativity, is based on the hypothesis that the velocity of the light c is constant. Invariance is a consequence of the velocity c being constant. Therefore it is important to see whether and how, in the case of the origin moving in a space of two or three dimensions, the velocity of light is constant.

We shall consider the case when the origin describes a plane curve—that is to say, when it moves in a space of two dimensions. In this case we can see whether the curvature has any effect on the velocity of light.

Let T and N be unit vectors taken respectively on the tangent and the normal to the curve described by the origin O' ; X', Y' the coordinates of a point P referred to the moving frame of origin O' , the expressions of X'

and Y' being those given in the generalized Lorentz transformation *. We shall have then

$$P = O' + \beta_1 X' T + \beta_1 Y' N. \quad . \quad . \quad . \quad (1)$$

Taking into account the motion of O' , of which u_1 and u_2 are the velocities along T and N respectively, t_1' and t_2' the times given by the watches whose centres move along T and N respectively, the above expression of P becomes

$$P = O' + \beta_1 (X' + u_1 t_1') T + \beta_2 (Y' + u_2 t_2') N. \quad . \quad (2)$$

We shall indicate with t the time given by the watch that moves with the motion of the origin along the curve, and is, therefore, at rest relatively to the origin. In the derivation of the quantities of the equation (2) we shall take as independent variable time, as expressed by t, t_1, t_2 , and the derivation of the dependent quantities will be made relatively to the time on which each quantity depends; we shall have then

$$\frac{dP}{dt} = \{\beta_1 (X' + u_1 t_1')\}' T' + \{\beta_2 (Y' + u_2 t_2')\}' N' \\ + \{\beta_1 (X' + u_1 t_1')\}' T + \{\beta_2 (Y' + u_2 t_2')\}' N, \quad (3)$$

in which

$$\{\beta_1 (X' + u_1 t_1')\}' = \frac{\frac{dX'}{dt_1'} + u_1}{1 + \frac{\frac{dX'}{dt_1'} u_1}{c^2}},$$

$$\{\beta_2 (Y' + u_2 t_2')\}' = \frac{\frac{dY'}{dt_2'} + u_2}{1 + \frac{\frac{dY'}{dt_2'} u_2}{c^2}},$$

$$T' = \frac{v}{\rho} N,$$

$$\text{and } N' = -\frac{v}{\rho} T,$$

* Phil. Mag., July 1934, p. 190.

for the formulæ of Frenet, $1/\rho$ being the curvature of the path and v the velocity of O' along the path. Therefore

$$\frac{dP}{dt} = \left\{ \begin{aligned} & \left[\frac{\frac{dX'}{dt_1'} + u_1}{1 + \frac{\frac{dX'}{dt_1'} u_1}{c^2}} - \beta_2 (Y' + u_2 t_2') \frac{v}{\rho} \right] T \\ & + \left[\frac{\frac{dY'}{dt_2'} + u_2}{1 + \frac{\frac{dY'}{dt_2'} u_2}{c^2}} + \beta_1 (X' + u_1 t_1') \frac{v}{\rho} \right] N. \end{aligned} \right\} \quad (4)$$

If the origin describes a straight line which coincides with the axis X , we have then $\rho = \infty$ and $Y' = 0$, and we find as a particular case of the above equation the formula given by the Relativity Theory for the expression of $\frac{dP}{dt}$. If to this hypothesis we add also $\frac{dX'}{dt_1'} = c$, we obtain $\frac{dP}{dt_1'} = c$, as it is known also from the Relativity Theory.

Let us examine a few cases which particularly depend on the value of the curvature $1/\rho$ of the path O' :—

(1) If $\frac{1}{\rho} = 0$ and a ray is sent from P , such that it reaches O' in the direction of either T or N , the velocity of the ray in either case is c .

(2) If the ray of light reaches O' in a direction which is neither that of T nor that of N , for the time being, then we may ask what would be the velocity of the components of the ray taken along T or N , in the hypothesis of $\frac{1}{\rho} = 0$ and the straight path of O' not being coincident with either T or N , but making with the direction of either of them an angle ϕ at the time of its arrival at the origin. In this case we have

$$\frac{dX'}{dt_1'} = c \sin \phi, \quad \frac{dY'}{dt_2'} = c \cos \phi,$$

and by substituting these values in (2) and replacing c

with the values $c \sin \phi$ and $c \cos \phi$ in the denominators of the same equation, we obtain

$$\text{mod. } \frac{dP}{dt} = c.$$

(3) If $\frac{1}{\rho} \neq 0$ and two rays start simultaneously from P with the velocity c , and reach the frame in the direction of T and N respectively * so that

$$\frac{dX'}{dt_1'} = c \text{ and } \frac{dY'}{dt_2'} = c, \text{ or } X' = ct_1' \text{ and } Y' = ct_2',$$

we shall have for the component velocities

$$c + \beta_1(X' + u_1 t_1') \frac{v}{\rho} = c + \beta_1(ct_1' + u_1 t_1') \frac{v}{\rho},$$

and putting $\beta_1(c + u_1) \frac{v}{\rho} = \gamma_1,$

we have $c + \gamma_1 t_1'$ along N ;

and putting $\beta_2(c + u_2) \frac{v}{\rho} = \gamma_2,$

we have $c - \gamma_2 t_2'$ along T.

If we assume as origin of the moving frame a point of the surface of the earth so that u_1 is approximately $=v$, and we make the initial velocity $c=0$, then $\gamma_1 = \frac{v^2}{\rho}$, which is the acceleration of a body starting with an initial velocity zero and falling towards the centre of the earth along the normal N.

The quantities γ_1 and γ_2 represent the effect of the velocity of the origin on the initial velocity c of the light.

The effect of the curvature of the path of the origin can be shown more directly by referring P to the frame of fixed origin O, which is assumed as the initial position of O' *. The axes T and N have then only a movement of rotation equal to the rotation of O' along the curve,

* Imagining that they belong to the same pulse of light, leaving P and spreading in all directions.

X and Y being the coordinates of P taken on such frame rotating round the centre O *. We have, then,

$$P = O + XT + YN ;$$

therefore

$$\begin{aligned} \frac{dP}{dt} &= XT' + YN' + X'T + Y'N \\ &= \left(X' + \frac{Yv}{\rho} \right) T + \left(Y' - \frac{Xv}{\rho} \right) N. \end{aligned}$$

X' and Y' being the derivatives of X and Y.

Therefore, in the case that the origin describes a curved path, the Lorentz transformation, in its original form, viz., based on the hypothesis that the origin moves along one of the axes, cannot show the properties of the motion of a point referred to a rotating frame. It served well in the approximate hypothesis of the motion of the earth being rectilinear and uniform †, during short intervals of time, sufficient to explain Michelson and Morley's result, but for the use of the transformation in the analysis of phenomena with frames moving with any motion the generalized transformation is necessary.

The analysis of motion with moving frames began with the restricted Relativity; the theory, however, did not go beyond the use of a frame whose origin moves with a linear and uniform motion, and turned to consider the possibility of expressing the laws of physics whose validity is independent of the choice of any fixed frame. The generalization of the first step, made by Relativity, might have been the study of phenomena with frames whose origin moves with any given law, since such are the frames which we can take on earth, the sun, or any planet. The expansion of Relativity in this direction is the object of this note and of those which precede it. Taking the earth as the origin of a frame, moving in a space of two dimensions, and assuming its motion as a *first datum*, we can see how limited appears some fundamental truth which Relativity assumed as universal, while remaining within the limits of a frame, whose origin moves along one direction only.

* Phil. Mag., July 1934, pp. 187, 188.

† See Chwolson (ed. 1914), vol. v. pp. 225 & 229.

LXV. *Some Quantitative and Qualitative Observations on the Electron Diffraction Pattern from the Natural (111) Face of Diamond.* By R. BEECHING *.

[Plate VIII.]

Introduction.

ELECTRON diffraction by reflexion from a single crystal cleavage face differs considerably from the analogous X-ray case, in so far as several diffracted spots may appear at once and the spots persist over quite a wide range of angle of incidence. Attention was first drawn to the latter phenomenon by Prof. G. P. Thomson, and it has been attributed to lack of penetration of the electron beam. The great objection to this explanation of the persistence of the spots is that the Kikuchi lines, from crystals showing considerable spread of the spots, are quite sharp. Since there is no reason for supposing that the diffuse radiation which gives rise to the lines will penetrate more than the main electron beam, it is difficult to account for the sharpness of these lines. We should expect their angular breadth to be of the same order as the range of angle of incidence over which the spots are found to persist.

The purpose of this work was to measure the angles over which the spots persist and also the breadth of the corresponding Kikuchi lines. The diamond was used because it very closely approaches a perfect crystal. This was confirmed by the fact that all the specimens used gave very sharp Kikuchi lines, and sharp lines are an indication of interior crystal perfection. Also, the diamond shows the persistence of the spots in a very marked manner. If an electron beam is diffracted by reflexion from a natural face of diamond which is parallel to the (111) planes a faint spot, in the position for specular reflexion, is visible at all angles of incidence. At the Bragg angle the spot becomes much more intense and then, as the angle is increased, it fades until it reaches a minimum at an angle about two-thirds of the way between one order and the next. As the angle is further increased the intensity again rises to a maximum value for the next Bragg reflexion.

* Communicated by Prof. G. P. Thomson, M.A., F.R.S.

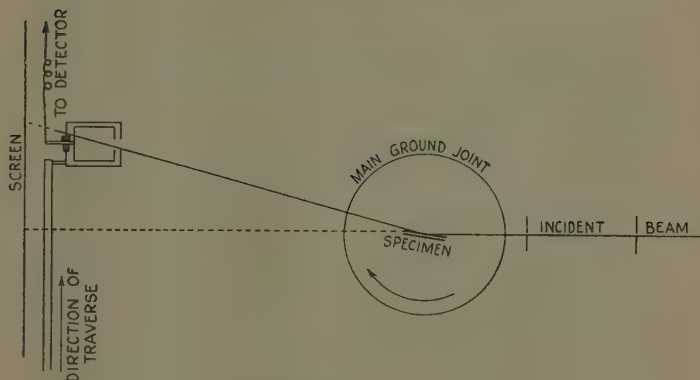
In this work also the intensity of the second order reflexion has been compared with that of the third order (the first order does not appear on account of the high inner potential of diamond). The second order should not appear according to the simple theory either in the case of X-ray or electron diffraction, since it is forbidden by the structure factor. A faint second order appears with X-rays, but in the case of electrons the second and third orders have comparable intensities, as Prof. G. P. Thomson has shown.

Apparatus and Experimental Procedure.

The electron diffraction apparatus used was a modification of that designed by Prof. G. P. Thomson. A small Faraday cylinder was arranged to traverse across the inside of the camera at right angles to the beam and just in front of the fluorescent screen. The cylinder was moved by means of a rack and pinion, worked through a ground-joint. The hole in the earthed casing of the cylinder was between 3 and 4 mm. in diameter, and that in the inside chamber was of the same size. The inner casing was insulated from the outer by an amber bush, and the lead from it was connected to a terminal passing through an insulating bush in the outer wall of the camera. The connexion between this terminal and the detecting system was made by a thick copper wire enclosed in a glass tube, and supported in an earthed brass tube by amber bushes. The current reaching the cylinder was allowed to leak away through a high resistance of the sputtered chromium type, of the order of 2×10^{11} ohm. The potential drop across this resistance was measured by a Lindemann electrometer, the circuit arrangement being that usually used with such instruments. The whole detecting system was included in an earthed, perforated zinc box, which provided satisfactory shielding in spite of the presence of the high tension supply in the room. Slow changes of the intensity of the primary beam and of the leakage resistance caused variation in the readings obtained from day to day, but since absolute measurements of intensity were not required this slow changing of conditions was of no importance. A more serious difficulty was the maintaining of the beam at a constant voltage during the course of a set of readings

which might require up to half an hour to obtain. The high tension supply circuit was the one usually employed in conjunction with the Thomson-Fraser type of apparatus, except that the capacity of the smoothing condenser was increased to $\cdot 014$ microfarads. The discharge was run for about half an hour before readings were commenced, and the spark-gap was adjusted so that a spark passed once or twice per minute. A fall of about 1 kv. in the voltage across the tube, due to pressure change, was accompanied by an increase of current which was easily detectable by means of a millimeter connected in the earth lead from the camera. An increase in voltage

Fig. 1.



Arrangement of apparatus.

caused a spark to pass, and with this arrangement a fairly steady potential drop was maintained across the tube. However, the greater part of the uncertainty in this work was due to variations in the voltage.

In order to ensure that the whole of the electron beam struck the crystal face, even at the smallest angles of incidence employed, the distance between the holes defining the beam was increased from the usual 6.5 to 11.5 cm. This brought the top hole, which, like the bottom hole, was 0.15 mm. in diameter, within 1.5 cm. of the specimen. This not only had the effect of reducing the divergence of the beam but also ensured that the beam was only about 0.15 mm. wide where it struck the specimen. The

specimen was approximately 1 cm. long, so that even at a glancing angle as small as $\cdot 02$ radians the whole beam struck the face.

The changes in angle of incidence of the beam were measured by means of a galvanometer lamp and scale, the spot being reflected from a small mirror on the main ground-joint of the camera, to which the specimen was attached.

The adjustment of the Faraday cylinder was made in the following way. The cylinder was pivoted eccentrically at the end of the traversing arm, so that by rotating it about this point it was possible to adjust it in a direction perpendicular to the line of traverse and to the beam. When the camera was in use the screen continued to fluoresce after the beam had been cut off. The receiver was adjusted so that the persisting image of the spot fell at the centre of the circular shadow of the cylinder, thus ensuring that the diffracted beam was entering the collector.

When the discharge was running steadily, having been adjusted as described, the angle of incidence was reduced to less than the Bragg angle for the second order, and the collector was adjusted to receive the faint reflected spot. The angle of incidence was then increased by stages by rotating the main ground-joint, and the intensity of the corresponding reflected spot was recorded. In order to eliminate the background another traverse was made along a line parallel to the centre line but about 3 to 4 mm. from it. This method gave a curve which was certainly a good approximation to the background distribution of intensity.

When calculating the spread of a spot by his dynamical theory, Bethe ⁽¹⁾ assumed that there was only one strong diffracted beam. In order to satisfy this condition as far as possible the azimuth setting of the crystal was always arranged so that any spots appearing during the traverse other than those in the plane of incidence were always far out from the central spot, since then the interaction is less important.

The Diamonds.

The diamonds used were kindly lent by Sir Robert Robertson, F.R.S. ⁽²⁾, and the numbers by which they are referred to here are those which he has given them.

Details about some of the stones will be found in his paper on the two types of diamond.

D. 2. "Extraordinary type." Most of the work was done with this stone, which had a highly polished (111) natural face showing only a few cleavage lines. The face was approximately an equilateral triangle of 1 cm. side.

D. 1. "Ordinary type." This stone had previously been examined by Prof. G. P. Thomson by electron diffraction, and had given a good pattern. When examined for the purpose of this work it was found to give only a few blurred spots and much background. This could probably be attributed to the fact that gold had been sputtered on to it since it was previously examined.

D. 10. "Ordinary type." This stone was smaller than the others used, having a natural (111) face which was approximately circular and about 4 mm. in diameter.

D 26. "Ordinary type." This stone had a natural (111) face which was an equilateral triangle of 1 cm. side. It charged up very badly, even when mounted in a cell with mercury backing.

D. 27. "Ordinary type." This diamond had two natural (111) faces. Examination under the microscope showed small equilateral triangular plates on these surfaces, presenting the appearance of parallel growth.

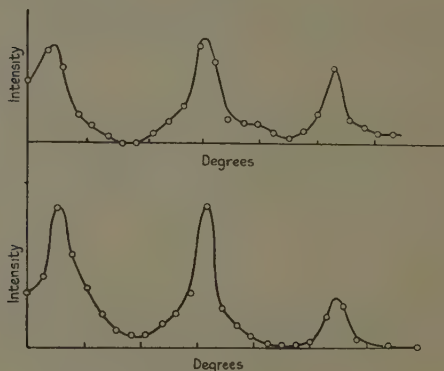
D. 28. "Ordinary type." This specimen had a triangular natural (111) face of about 1 cm. side. Microscopic examination showed it to be dotted with small pyramid growths. The stone did not give a very good pattern.

Results of Intensity Measurements.

From the sets of readings obtained curves were plotted showing intensity in arbitrary units against change in angle. The curve for the background intensity was plotted on the same graph, and from these pairs of curves, curves showing the difference in intensity between the spots and the background were plotted. Two typical curves of this sort are shown in fig. 2, while the relative intensities and half intensity spreads of the various orders obtained from five curves are shown in Table I. The values of the spread shown in the table are the sum

of the deviations on either side of the Bragg position, which reduce the intensity to half the maximum.

Fig. 2.



Curves from D. 2.

TABLE I.

D. 2.

No.	Wave-length.	Half-intensity spread in minutes of arc.			Relative intensities.	
		2nd order.	3rd order.	4th order.	3rd order 2nd order	3rd order 4th order
1.....	·067 Å.	33'	23'	18'	.77	2·6
2.....	·072	43	29	19·5	1·12	1·39
3.....	·060	38	36·5	25	1·26	1·32
4.....	·069	23	29·5	23	1·46	3·95
5.....	·072	38	21	23	1·02	2·58
Mean ..	—	35±3	28±3	22±1	1·1±·1	2·4±·5

The mean spreads are shown, although all the curves were not taken at the same voltage. This has been done since the change of spread over the range of wave-length used is likely to be less than the experimental error.

A number of preliminary curves were obtained with the apparatus before the anode was lengthened and with

only $\cdot 002$ microfarads smoothing in the high tension supply circuit. The values for the relative intensities of the orders agreed very well with those obtained under the final conditions, but the spreads were 20–30 per cent. larger. This was probably due to slight inhomogeneity of the beam. A few curves were taken under the original conditions with D. 10. The curves were of the same general shape as those with D. 2 except that the second order was about 20 per cent. less intense and its spread uncertain, due to charging up.

Examination of the Kikuchi Lines.

In the first case an attempt was made to measure the intensity distribution across the Kikuchi lines by means of a Faraday cylinder with a fine slit in it parallel to the line. It was found possible to detect the lines in this way, but owing to the fact that they were only about 5 per cent. more intense than the background it was impossible to measure the distribution of intensity. This method was given up in favour of a photometric method.

For this purpose the plate attachment, which had been removed for the first part of the work, was replaced. The part of the plate to be exposed at any one time was limited, by means of a mask of sheet brass, to a strip 3 mm. wide perpendicular to the shadow edge (*i. e.*, to the line in which the plane of the crystal met the screen) and slightly to one side of the centre line of the pattern. The specimen was adjusted so that several orders of the lines parallel to the shadow edge were seen crossing the strip. A number of exposures were then made on one plate, the time of the first exposure being one second and those of the succeeding exposures increasing by intervals of one second. From seven to eleven exposures were made on each plate, the strips being parallel side by side.

In order to obtain intensity curves from the photographic records the plates were microphotometered by means of a Cambridge recording microphotometer. A traverse was made along the one-second exposure strip and a second traverse was made at right angles to the strips. From the second record, which presented the appearance of a series of flat-topped humps, a curve

was plotted showing the instrument reading against the corresponding value of It , where I is the electron intensity, which was the same at corresponding points on all the strips, and t the time of exposure. It has been shown that the blackening of the plate by electrons depends only on the product It , and it was therefore possible by means of this curve to convert the photometer record for the traverse along the strip into a curve showing intensities against distance along the strip.

Three curves were obtained in this way for D. 2 and two for D. 27.

Results of Kikuchi Line Measurements.

Two typical intensity curves are shown in fig. 3. The intensity in arbitrary units is shown plotted against

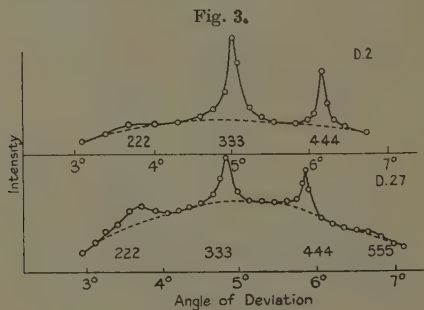


TABLE II.

Curve.	Wave-length.	Angle of half-intensity spread in minutes.		
		2nd order.	3rd order.	4th order.
D. 2. Curve 1.....	·071 Å.	15'	9'	5·5'
2.....	·071	—	11	7·5
3.....	·072	20	10	6·5
D. 27. Curve 1.....	·061	16·5	10·5	5·5
2.....	·067	—	8	7·5
Mean	—	17·2	9·7	6·1

the angle between the main and the scattered rays. The half-intensity spreads of the lines are shown plotted in

Table II. The relative intensities of the lines depend upon the distribution of intensity in the diffuse radiation producing them, and hence upon their proximity to a diffracted spot. For this reason values of the relative intensities of the Kikuchi lines have little meaning until more is known about the diffuse radiation.

These results should all be reduced by about 1.5' to allow for the finite width of the beam.

Determination of the Inner Potential.

A number of determinations of the inner potential were made for D. 2 and D. 26 by the rotation method as first used by Yamaguti ⁽³⁾. The necessary measurements were made on the fluorescent screen. The inner potential was also determined in the case of D. 2 and D. 27 from measurements on the Kikuchi lines parallel to the shadow edge, the method being due to Shinohara ⁽⁴⁾. The measurements for this purpose were obtained from photographs.

Results of Inner Potential Measurements.

The values of the inner potential for the various specimens examined are shown tabulated below.

TABLE III.

" Ordinary type."		" Extraordinary type."	
Rotation method.	Kikuchi lines.	Rotation method.	Kikuchi lines.
volts.	volts.	volts.	volts.
D. 26. 24.3	D. 27. 20.1	D. 2. 23.4	D. 2. 20.7
20.4	17.9	19.4	17.5
24.0	20.7	21.4	21.0
	20.7	19.4	20.6
Mean 6.. $21.1 \pm .8$ volts.		Mean ... $20.4 \pm .6$ volts.	

The mean value for all specimens is $20.8 \pm .7$ volts.

Qualitative Examination of the Diamond Pattern.

A number of ordinary electron diffraction photographs were taken from D. 2, D. 27, and D. 28; also rotation

photographs were taken with D. 2 and D. 27. The latter specimen charged up slightly at small angles of incidence, but the plates showed the persistence of the spots over a considerable range of angle of incidence and also the asymmetry of spread mentioned in the Introduction.

In the case of D. 2 it was found that for all settings of the azimuth a faint fine line, similar to a Kikuchi line, always appeared, passing through the spot in the plane of incidence and perpendicular to the shadow edge. In symmetrical or nearly symmetrical settings of the crystal as many as five similar vertical lines appeared, passing through the various side spots. In such settings, the lines were much more intense (figs. 5 & 6, Pl. VIII.). The broad bands which are normally found with other crystals under such circumstances were not observed.

It has not been decided to what these lines are due. With many crystals elongation of the spots in a direction perpendicular to the shadow edge is observed, and may usually be attributed to one of two causes :

- (1) Part of the crystal surface being broken into small blocks, tilted slightly with respect to one another.
- (2) Part of the crystal surface being composed of very small blocks which have maintained their alignment but which are separated by fine cracks.

The effect of crystal imperfection of the first type would be to produce a rotation pattern superimposed upon the ordinary spot pattern. The lines should therefore show minima at certain places, but they do not. The elongated spots produced should be rather broad, since the normals to the small crystal facets will not necessarily lie in the main plane of incidence. Actually the lines are very sharp; also this theory would not account for the greatly increased intensity of the lines in all symmetrical settings of the azimuth.

The objections to an explanation based upon a supposed imperfection of the second type are very similar. The way in which elongated spots are produced by such imperfection is by relaxation of one of the diffraction conditions described by Kirchner ⁽⁵⁾, as a result of the small extent of the crystal blocks in the direction of the beam. The pattern produced will resemble a rotation picture superimposed on the ordinary pattern. As has already been

pointed out the lines from diamond did not show the features to be expected in a rotation pattern.

For the reasons given it seems that the observed lines cannot be attributed to the causes mentioned. Many crystals, when set in symmetrical or nearly symmetrical positions with respect to the electron beam, give rise to a pattern of dark bands, an effect first mentioned by Shinohara ⁽⁶⁾. No such bands were observed in the case of diamond, and it is possible that the vertical lines are a modified form of band. In some cases a vertical white line also appeared; one such is just visible inside the bright line on the left of fig. 6 (Pl. VIII.).

With all the specimens which gave good patterns it was possible to find a symmetrical setting, such that the fourth order reflected spot was flanked on either side by an intersection of several Kikuchi lines. Under these circumstances the spot was noticeably diminished in intensity by the proximity of the lines. This effect is shown in figs. 7 and 8 (Pl. VIII.); fig. 7 shows the general appearance of the pattern, while fig. 8 shows the spot when normal and when partially suppressed. The two exposures were made on the same plate for the same time.

This suppression of the spot would seem to indicate interference between the spots and the Kikuchi lines. If this is so it must mean that the lines are produced, partly at least, by electrons which have only been scattered elastically.

Discussion and Comparison with Theory.

This work has shown no definite differences between the electron diffraction patterns obtained from the two types of diamond. It is felt that no importance can be attached to the fact that the second order reflexion appeared to be slightly less intense in the case of the ordinary type, since the specimen used charged up badly. All the specimens of the ordinary type showed this charging up, while D. 2 only charged at very small angles of incidence indeed. Since D. 2 was the only stone of the ordinary type examined it is impossible to decide whether this difference is a feature of the two types or not. D. 2 showed the vertical lines in a much more

marked manner than the other specimens, although D. 27 showed the effect slightly. With none of the specimens were any bands observed.

The quantitative results are not in agreement with any existing theory. Bethe's ⁽¹⁾ dynamical theory predicts that over a small range of angle the reflexion will be *total*, while outside this range the intensity will fall off very rapidly. In the case of electrons of the velocities used the range of total reflexion predicted is of the order of $2'$ for the (333) reflexion, and less for higher orders. Actually the diffracted spots are found to have an intensity of only about 1 per cent. of that of the primary beam, and the third order has a half-intensity spread of about $30'$.

Recently Kikuchi ⁽⁷⁾ has arrived at a result similar to that of Bethe by a somewhat different method. He has also measured the spread of the spots for zinc blende by photometric measurements on a rotation photograph, and he finds that the actual spreads are about twenty times those predicted by theory. This is in good agreement with the results obtained from diamond. Kikuchi suggests that the discrepancy between the theoretical and practical results is due to the fact that absorption has not been taken into account in the theory, while in practice it limits the depth of penetration of the electron beam. As has been pointed out in the Introduction this is the view that has been advanced many times before to explain the spread of the spots, but it does not account for the sharpness of the Kikuchi lines.

From the thickness of polycrystalline metal film which an electron beam will penetrate before falling to half intensity (*i. e.*, about $500 \text{ \AA}$.) it is possible to deduce that at the small glancing angles employed in reflexion work the effective depth of penetration of the beam will be of the order of five or six planes. This is sometimes advanced as an argument in favour of the theory that the spread is due to lack of penetration, and since the observed form of curve would be produced if only about four planes were effective the agreement seems to be good. It is questionable, however, whether the value of penetration, determined on the basis of transmission through polycrystalline films, may be applied to the case of reflexion from a single crystal at glancing incidence.

In the case of reflexion from some types of crystal a so-called "surface pattern" occurs, consisting of two or more semicircles of spots which occur simultaneously and persist over a wide range of angle of incidence. Such patterns satisfy the condition for diffraction by a plane cross-grating (as formed by the top net plane) at grazing incidence. The appearance of such patterns is usually explained in terms of lack of penetration. This type of pattern might, however, be equally well explained by supposing that the top few planes differ from the bulk of the crystal in some respect.

J. W. Harding has suggested that the observed effects may be due to an expansion or contraction of the spacing between the surface planes. He is at present working further upon this idea, and until the theory has been fully considered it is impossible to say whether it will explain the experimental results, but it promises to do so.

The appearance of the (222) reflexion is forbidden on the simple theory, but Bethe's dynamical theory predicts such a beam as a result of interaction between the (111) and (333) beams in the crystal. Bethe⁽¹⁾ has suggested that the weak (222) beam found with X-rays may be due to a similar effect made possible by the high perfection of the diamond. It has been shown that the relative intensity of the (222) reflexion is less the harder the X-rays and the less perfect the diamond, which suggests that it is dependent upon the magnitude of the interaction between the waves and the crystal. In the case of electron waves, which are easily scattered, we should expect the relative intensity of the forbidden order to be far greater than in the case of X-rays.

A commonly accepted explanation of the appearance of a (222) order from diamond with X-rays is that the carbon atoms are distorted by their neighbours and can therefore no longer be regarded as spherically symmetrical as they are in the simple theory. Since in the case of electron waves the greater part of the scattering is due to the nucleus, we should expect the effect of such distortion to be less than for X-rays, since it is likely to affect only the outer shells of the atom. It has already been pointed out that the (222) order is relatively far more intense in the case of electrons than for X-rays, so that it seems highly improbable that the appearance of the forbidden orders can be attributed to distortion in the

electron case. The second order cannot be due to a purely surface effect, since a fairly sharp (222) Kikuchi line appears. It would seem probable therefore that the forbidden maximum is produced in the manner suggested by Bethe.

Attention has been drawn in the Introduction to the fact that the intensity of the reflected spots does not fall off symmetrically on either side of the Bragg angle. The intensity falls off more slowly on the side of large angle, which is in the opposite sense to the asymmetry observed by Yamaguti ⁽⁸⁾ in the case of a natural face of pyrite.

Kikuchi has offered an explanation of the effect observed by Yamaguti ⁽⁸⁾. He attributes the asymmetry to the effect of intermediate crystal planes. He considers first of all a simple crystal with equally spaced planes parallel to the surface, and by assuming that each plane scatters the same amount of the incident wave he arrives at the solution

$$|S_0|^2 = \frac{\sin^2\left(\frac{2\pi}{\lambda} nd \sin \theta\right)}{\sin^2\left(\frac{2\pi}{\lambda} d \sin \theta\right)},$$

where $|S_0|^2$ is the intensity of the scattered beam, n the number of planes, and d the plane spacing.

In the case where there are intermediate planes distance x from the original planes this expression becomes

$$|S|^2 = 4S_0^2 \cos^2\left(\frac{2\pi}{\lambda} x \sin \theta\right),$$

where $|S|^2$ is the new intensity of the scattered beam and S_0 has the same meaning as before. Since $|S_0|^2$ represents a symmetrical distribution in the spots the asymmetry is introduced by the $\cos^2\left(\frac{2\pi}{\lambda} x \sin \theta\right)$ term.

In the case of diamond x is $d/4$, so that this term becomes

$$\cos^2\left(\frac{n\pi}{4}\right).$$

On this theory the second order is missing, the third order falls off more slowly on the side of large angle, the fourth order is symmetrical, while the fifth order falls off more slowly on the side of small angle. Actually all the orders fall more slowly on the side of large angle, so that this theory certainly does not hold in the case of diamond.

Summary.

Intensity measurements have been made on the electron diffraction pattern obtained from a (111) face of diamond. The relative intensities of the main spots have been determined and the tolerances in the Bragg angles have been compared with the widths of the corresponding Kikuchi lines. The inner potential has been measured and various qualitative observations have been made. A case where a spot has been partially suppressed by Kikuchi lines is recorded.

In conclusion, I should like to thank Prof. G. P. Thomson for suggesting the main problem and for his helpful advice and kind interest during the work; I should like also to thank Sir Robert Robertson for the loan of the diamonds which has made this work possible.

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Imperial College of Science, S. Kensington.
21st June, 1935.

LXVI. *Magnetic Birefringence in Solutions of Paramagnetic Salts of Rare Earths.* By S. W. CHINCHALKAR, D.Sc., Indian Association for the Cultivation of Science, Calcutta *.

IN a recent paper † on "A new Type of Magnetic Birefringence" the author gave an account of some measurements on the magnetic birefringence exhibited by aqueous solutions of paramagnetic salts of rare earths. One of the conclusions drawn therein was that the paramagnetic ions should possess an electronic orbital moment in order that they may exhibit the phenomenon. The main basis for this conclusion was that the Gd^{+3} ions, which are in the S-state ($^8S_{7/2}$), do not develop any birefringence in the magnetic field. A parallelism was traced in the paper between the magnetic birefringence and the paramagnetic Faraday effect exhibited by solutions of rare-earth salts, both of which phenomena depend ultimately on the orientations of the paramagnetic ions in the magnetic field. Solutions of gadolinium nitrate did not naturally show either phenomenon to an appreciable extent. (The paramagnetic Faraday effect observed in xenotime crystal, which was regarded by Kramers and Becquerel ‡ as due probably to the Gd^{+3} ions, is really to be attributed, as Becquerel and de Haas § have shown later, to the Tm^{+3} ions, which also are present in xenotime, and which, being in the $^3H_6$ state, possess a large orbital moment.)

Recently Haenny * has published the values of magnetic birefringence for practically the whole series of rare earths. His values for the solutions of different salts refer to different concentrations, and for intercomparison it is desirable to reduce them to the same equivalent concentration. In the following table Haenny's results have been reduced to concentrations equivalent to that of the Yb nitrate solution employed by him, assuming for the purpose of this calculation that the variation of magnetic birefringence with concentration is given by the linear law.

* Communicated by Prof. K. S. Krishnan, D.Sc.

† Ind. Journ. Phys. vi. p. 581 (1931).

‡ Proc. Amst. Acad. xxxii. p. 1190 (1929).

§ Journ. de Phys. iii. p. 109 S (1932) (Bull. Soc. Franc. de phys. no. 325).

¶ Compt. Rend. cxcv. p. 219 (1932).

It is known, however, that the birefringence does not vary linearly with the concentration, though the deviations from the linear law are not large. For example, for solutions of cerous nitrate it is known that with increasing concentration the value of the magnetic birefringence increases more than is indicated by a linear law, and this is probably true of solutions of other salts as well. The value given in the table will, therefore, be numerically (not algebraically) too high or too low according as the particular concentration from which the value given in the table has been calculated is higher or lower than the standard concentration adopted. These uncertainties are

Atomic no. of element.	Ions in solution.	Birefringence (relative to nitrobenzene = 100).	"L" of the trivalent ion.
58	Ce ⁺³	< -17.2	3
59	Pr ⁺³	< - 7.4	5
60	Nd ⁺³	0	6
61	Il ⁺³		6
62	Sm ⁺³	< - 2.0	5
63	Eu ⁺³	> + 5.9	3
64	Gd ⁺³	0	0
65	Tb ⁺³	> -82.3	3
66	Dy ⁺³	> -10.3	5
67	Ho ⁺³	> -74.0	6
68	Er ⁺³	> +35.7	6
69	Tm ⁺³		5
70	Yb ⁺³	+53.5	3

indicated by suitable symbols, | < | or | > |, prefixed to the figures, which respectively indicate that the probable values are *numerically* less than or greater than those entered in the table.

An inspection of the values for birefringence reveals an interesting regularity. It is known that the rare earth series can be divided into two halves, the first (up to Gd) corresponding to the erect and the second to the inverted multiplet intervals. In both the halves the orbital moment of the ions, according to Hund's theory, undergoes a similar variation, the "L" values being as shown in the table. It is interesting to find that in each half the

magnetic birefringence has a large negative value for the first member (Ce and Tb), becomes less in magnitude for successive members, and ultimately becomes positive towards the end of each half (Eu, Yb). (There is an irregularity at Nd and Dy, both of which have a feeble birefringence.) This confirms the connexion between the birefringence and the orbital moment of the ion.

The change of sign reminds one of the negative sign of the electro-optic Kerr effect in some fluids containing polar molecules, which has been explained as due to the direction of the dipole moment being nearly perpendicular to the direction of the maximum optical polarizability of the molecule *. If the applicability of the Langevin theory to the rare earth solutions as given in the author's paper † be correct, the above variations and the change of sign mean that the angle between the magnetic moment of the ion and the direction of its maximum optical polarizability varies as the different 4f electrons are added to form the successive members in each half of the rare earth series.

LXVII. *Notices respecting New Books.*

Cycles of Reduced Ideals in Quadratic Fields. By E. L. INCE.
[Pp. xvi+80.] (Office of the British Association : London, 1934.)

THIS is the second volume of tables published by the British Association by means of the bequest of Lt.-Col. A. J. C. Cuninghame. The appeal is to so small a body of mathematicians that publication would have been unlikely without such financial assistance; the few—of every nationality—who are interested in the theory of algebraic numbers will welcome the circumstances which have made it possible.

Any ideal in a quadratic field is equivalent to a *reduced* ideal, in which the members of the basis satisfy certain inequalities. It is easily seen that the number of reduced ideals in a field is finite, and that they may be arranged in cycles. The tables give the reduced ideals for every real quadratic field ($\sqrt{m}$) in which the non-square integer m is less than 2025. The arrangement is convenient and the printing clear.

* C. V. Raman and K. S. Krishnan, *Phil. Mag.* iii. p. 721 (1927).

† *Ind. Journ. Phys.* vi. p. 581 (1931).

New Pathways in Science. By Sir ARTHUR EDDINGTON.
[Pp. x+333.] (Cambridge University Press, 1935. Price
10s. 6d.)

THIS book, based on a series of lectures given at Cornell University in 1934, deals with a variety of topics, and is perhaps less of an artistic whole than Eddington's previous books, appearing rather as an addendum to them. In his preface Eddington states: "I was determined to make what I could of whatever had come to my mind in the last six years." There is, then, a unity, in that the various topics treated link into a comprehensive outlook on the physical and extra-physical world; and few scientists have so strongly developed a sense of "the interrelatedness of things" as Eddington.

In the first chapter Eddington summarizes his views on the relation between the world of physical science and the world of the "familiar story." The *dramatis personæ* are then introduced—electrons, protons, and so on of the old picture, "waves of fog" of the new. The next four chapters are connected technically as being based on statistical considerations. They deal with the end of the world, the determinacy problem, and probability. A survey of present views on the constitution of stars follows, and leads to a discussion of subatomic energy, in which are brought together the tentative conclusions drawn from astrophysical and laboratory evidence. There is an admirable chapter on the cosmic cloud, and the expanding universe is clearly and straightforwardly discussed. An exposition is then given of Eddington's treatment of the constants of nature, an exposition which, in its non-technical guise, carries complete conviction as to the soundness of the kind of argument which is used. This is virtually the climax of the book. An excellent chapter follows showing the kind of problem dealt with in the theory of groups and the reasons for the fundamental importance of the theory.

A suave but often devastating reply to various criticisms of philosophical aspects of Eddington's previous writings makes a chapter which is almost too readable. It reveals how easy it is to skate on the surface of some of the profounder questions, and how meretricious are some of the gyrations performed.

An implicit defence is also given, if defence were needed, of popular expositions of physical science; and the point is made that criticism of the scientific conclusions reached should be based on a study of the technical arguments. This is indisputable; but it raises a further point. In a book of this kind the material is not provided on which a criticism of such matters as the treatment of the constants of nature could be based; the would-be critic must study the original

papers. Those few experts who can appreciate these papers are by no means unanimous in their support of the conclusions. It is certainly most interesting to read, in the popular exposition, of the kind of way in which the conclusions have been reached. Almost, the average physicist understands it all, and, with the inviting introduction, he turns to the technical papers—and finds that it is only a very small stile that he has been helped over. In a field such as this, whose importance is widely recognized, would not an exposition for physicists, other than the few experts, such as Eddington has given in other fields, be of much greater present value than the intriguing, but essentially unsatisfying, exposition which is given here?

The final chapter, the epilogue, is a sincere, restrained, and beautifully written confession of faith built round the theme, "We are that which asks the question. Whatever else there may be in our nature, responsibility towards truth is one of its attributes."

It is unnecessary to say that this is a book which should be read by all those who have the slightest interest in physical science, whether in itself or in its wider bearings. Expert and non-expert alike will appreciate it for its clear exposition of the results of recent scientific work, its record of explorations in progress, its felicity of phrase and quotation, and its presentation of the outlook of one of the few among the greater men of science of to-day who has a clearly felt general outlook or the courage to express it.

Papers and Discussions, International Conference on Physics, London, 1934.

THESE highly successful and interesting meetings were held, as every one knows, in October 1934 under the joint Presidency of Professor R. A. Millikan and Lord Rayleigh. What more can be said other than that the distinguished contributors to these important volumes have recorded faithfully and well the substance of their deliberations over the few days concerned?

Congratulations are due to all concerned on this excellent permanent account of matters of real import in Physics at the present time. Printed by the Cambridge University Press for the Physical Society the three volumes are uniformly well produced in standard style, and in this respect are beyond criticism.

Vol. I. *Nuclear Physics*. [Pp. 257+figures.] (Price 10s.)

The address of welcome by Sir F. Gowland Hopkins, followed by the opening survey of Lord Rutherford, bring the reader to page 19, where the more detailed papers begin.

The topics include one paper on Quantum Theory; four on Natural Decay; three on Artificial Radio-Activity; five on Disintegration and Synthesis of Nuclei; eight on Cosmic Radiation. These papers, with the relevant discussions, present a clear account of our present state of knowledge of these subjects.

It would seem unnecessary to record here the names of the distinguished workers in these fields whose contributions are included.

Vol. II. *The Solid State of Matter.* [Pp. 183+figures.] (Price 10s.)

The opening survey by Sir William Bragg takes the reader to page 6. The thirteen papers which follow are distributed between three main groups: The Structure of Molecules and of the Ideal Lattice; the Deviations of Real Crystals from the Ideal Lattice Structure; Plasticity and Strain Hardening in Crystals.

The papers, needless to say, are of the same very high order as those in Vol. I., and again it seems unnecessary to append a list of the distinguished contributors of papers and discussions.

Vol. III. *Reports on Symbols, Units, and Nomenclature.* [Pp. 40.] (Price 2s. 9d.)

This most useful volume is an account of some of the proceedings of the general assembly of the International Union of Pure and Applied Physics held in London on October 5th, 1934. Professor Millikan was in the chair.

Four reports, under the ægis of Sir R. T. Glazebrook, President of the Commission on Symbols and Units, are here produced—in so far as their conclusions are concerned—dealing with:

1. The Standard Thermal Unit.
2. Electrical and Magnetic Units.
3. Certain Thermodynamic Symbols.
4. The future work of the Symbols, Units, and Nomenclature Commission.

It is interesting to notice that under 4 the consideration of terms in Optics and Acoustics is foreshadowed. The present writer's opinion is that particularly in technical Acoustics is this necessary. It is a new and rapidly grown subject, on the technical side especially, and a new terminology is in the making. Immediate attention may save much trouble in the future.

Electromagnetism. H. M. MACDONALD. [Pp. xv+178.] (G. Bell & Sons, Ltd., London, 1934. Price 12s. 6d. net.)

THE author's aim is the development of a consistent scheme for the presentation of electrical phenomena based on the fundamental laws of electromagnetism. He recalls Faraday's insistence on distinguishing between phenomena which are independent of the nature of material media and those which depend essentially upon the nature of the material. The laws governing the propagation of electrical effects in free space are developed by adopting the ampère and Faraday laws and Fresnel's law of transversality, and the author shows it is unnecessary to associate the concepts of dielectric constant and magnetic permeability with space which is devoid of matter. The presence of material media modifies the mode of propagation in such media, and account is taken of this by assuming a distribution of electric and magnetic current throughout the space occupied by matter.

In applications to transparent and conducting media, diffraction, radiation, resonance, and to moving electric systems, a number of selected problems are treated in detail.

Alchemy, Child of Greek Philosophy. By ARTHUR JOHN HOPKINS. [Pp. xi+262.] (New York: Columbia University Press; London: Humphrey Milford, 1934. Price 17s. 6d.)

THE title specifies one parent of alchemy; in the text Egyptian metallurgy is a second parent. It seems clear, through many uncertainties, that this infant chemistry was the child of Greek philosophy and Egyptian craftsmanship. Alexandria was probably its cradle, and the early Christian era was probably its birth-date.

Professor Hopkins, acknowledging an early stimulus from Berthelot, provides an interesting and able clarifying step in the precarious piecing together of alchemy from scattered data. He describes how the Egyptian metallurgical effort applied "Greek philosophy to metallic chemistry" by conforming its initial alloy, the "tetrasomy," as closely as possible to the "ideal 'matter' of Plato," and by imposing appropriate qualities serially upon it. He notes, among other debts to Aristotle, the familiar alchemistic attempt to add impetus to the tendency of metals towards their own perfection in gold. He also touches on the mingling of religious actions into alchemy.

The production of alloys coloured like silver or gold led to charlatanry; it also led to the belief that the baser metals could be transmuted into the nobler. According to Professor Hopkins, in gold-coloured alloys "Egyptian or true alchemy" did actually produce gold as it was originally "defined by the

alchemist." He insists that speculation passed from the "essential colouring principle of a dye," through colours imparted to metals and ideas of "ferments," to the potency of the Philosopher's Stone. The sources of the sulphur-mercury constitution of metals in Islamic alchemy are clearly exposed, and the Geber controversy, with its latest phases, is interestingly discussed.

The insistence on a Pseudo-Alchemy, transitional between Islamic alchemy and modern chemistry, may obscure one item. Boyle's tentative explanation of transmuting powers in the "powder of projection" and the alkahest, by applying the Cartesian corpuscular mechanism, is the seventeenth century analogue of applied Neoplatonism: alchemy always applied a philosophy to a lively sense of transmutational possibilities. Perhaps the final chapter on "The Phlogiston Theory" underestimates the phlogistian's power to accommodate even a loss of phlogiston to quantitative data. The book, however, is an able and attractive addition to the history of science.

Orthohydrogen, Parahydrogen, and Heavy Hydrogen. By ADALBERT FARKAS. [Pp. xiv+215.] (Cambridge University Press, 1935. Price 12s. 6d.)

SINCE the definitive experiments of Bonhoeffer and Harteck in 1929 showed that ordinary hydrogen was a mixture of two gases, and the spectroscopic investigations of Urey, Brickwedde, and Murphy in 1931 confirmed the existence of heavy hydrogen, an enormous amount of work has been carried out on what was previously considered a simple element. A new field for chemical investigation has been opened up, which is of particular interest owing to its necessarily close connexion with theoretical and experimental work on the more physical side. In this book Farkas, who has himself contributed largely to the progress that has been made, has given a well documented survey of the work which has been done.

In the first part, after an introductory account of the discovery of ortho- and parahydrogen, the theory is outlined and the relevant formulæ are stated and supplemented by numerical tables. The preparation and properties are then described, and there is an excellent chapter on the kinetics of the ortho-para-hydrogen conversion. Brief chapters follow on applications and on ortho- and para-systems in other molecules.

The second part, on heavy hydrogen, is on similar lines. The preparation and methods of estimation of heavy hydrogen are discussed in detail. In dealing with the properties an account is included of transmutations produced by deuterons, and there are well illustrated sections on the spectra. As in the first part, questions connected with entropy, free energy,

and chemical constants are fully treated. The final chapter deals with various equilibria and the kinetics of a number of reactions.

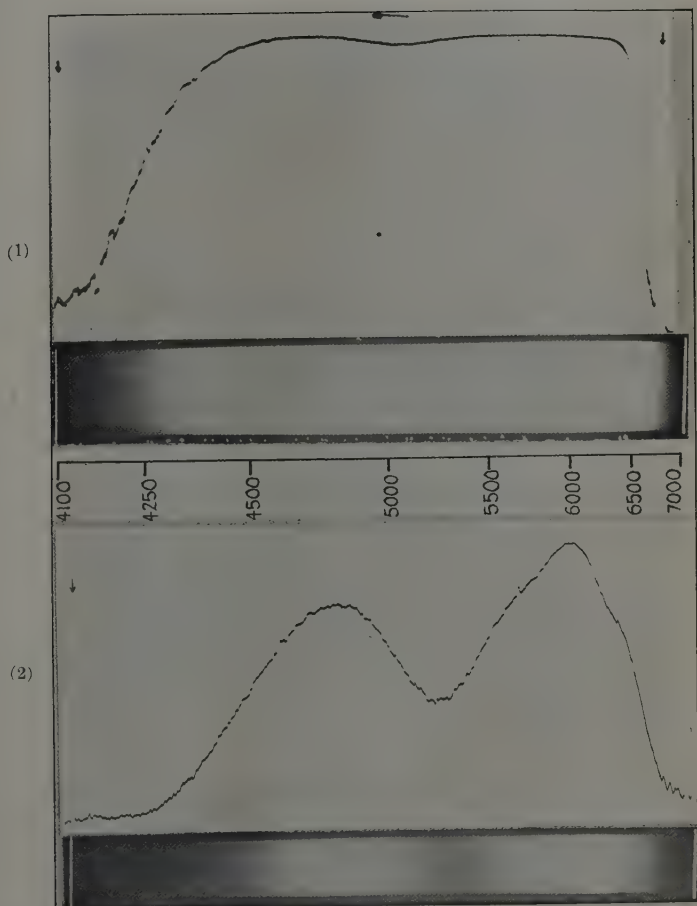
There is a bibliography at the end of nearly three hundred references, which are referred to in the text. The diagrams are excellent and the tables extensive. Some of these would be more immediately useful if accompanied by a descriptive title (rather than a description in the text). Very oddly, the book has no index. This is the more surprising as the primary value of the book will be for reference purposes rather than for continuous reading. The arrangement of the book, however, is good, and readers who are workers in the field will have little difficulty in finding what they require. The book is a first rate report on the present state of the subject by one who has made himself a master of the many interrelated aspects.

Sound. By E. G. RICHARDSON. (Second Edition.) [Pp. vii+319.] (Edward Arnold & Co., Ltd. Price 15s.)

THIS edition brings up to date Dr. Richardson's well-known text-book on Sound by including work done since the publication of the first edition in 1927. Three entirely new chapters are added on impedance, supersonics, and the reproduction of sound. The theory of acoustic impedance and of acoustic filters is clearly and concisely given; the reviewer would have welcomed a short account of the design and dimensions of acoustic filters. In the chapter on supersonics the recent work on the diffraction of light by supersonic waves is described. Throughout the book a happy balance is preserved between the mathematical and experimental sides of the subject and between its purely scientific interest and its many practical applications.

The book is an excellent one for university students, for whom it is primarily intended, but also contains much that will be of interest to technical and research workers in this field; those interested in the subject from the musical standpoint will find numerous references to musical instruments, scales, harmony, etc. The book is well printed and well bound, is attractively written, and has an adequate name and subject index. A very useful feature is the large number of references given in foot-notes; many more references are given than one usually finds in a text-book intended for students, but this will undoubtedly increase its value for other workers who may wish to pursue particular topics in greater detail. In all respects the book is very attractive, and can be thoroughly recommended.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]



Spectra of afterglow.

(1) Air.

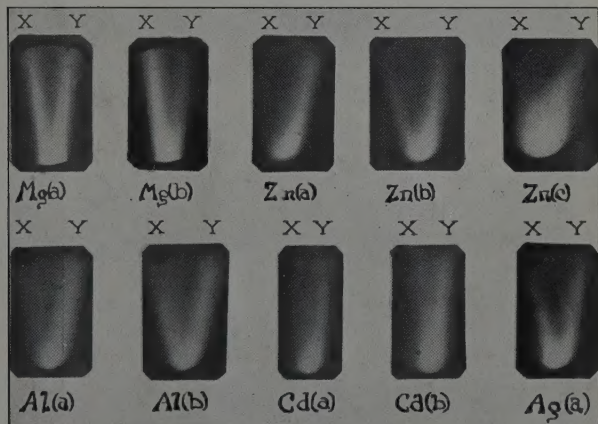
(2) Oxygen.

FIG. 3.



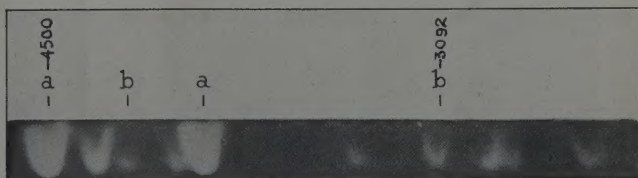
Spectrum of magnesium streamers showing the relative duration of the various lines.

FIG. 4.



Effect of inductance on magnetically deviated streamers: (a) very small inductance; (b) increased inductance; (c) large inductance.

FIG. 5.



Spectrum of aluminium streamers in a transverse magnetic field.



FIG. 4.



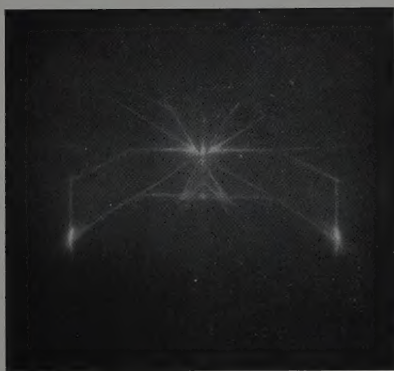
D. 27.

FIG. 5.



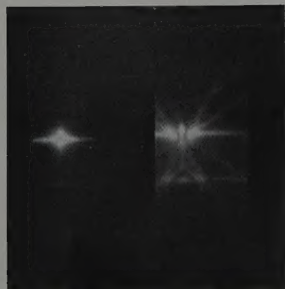
D. 2.

FIG. 7.



D. 2,

FIG. 8.



D. 2.

STATION 59